Affiliated Common Value Auctions with Costly Entry

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April 24, 2023

Abstract

We analyze costly participation in a first-price auction with affiliated common values and a large pool of potential bidders. We focus on auctions where the realized number of bidders is unknown at the bidding stage. In contrast to the standard case, both participation and bidding decisions are often non-monotonic in the symmetric equilibrium of our model. The expected revenue to the seller is often higher in the auction where the realized number of participating bidders is not disclosed. JEL Classification: D44

1 Introduction

In many markets, potential trading partners are dispersed and hard to identify ex-ante. An important function of the trading mechanism is to bring the different parties together so that gains from trade can be realized. Procurement auctions are organized to attract bids from potential service providers. A start-up can signal its willingness to be bought by one of many established firms in the industry. A firm may advertise a vacancy to solicit applications from potential employees. These situations can be broadly understood as auctions, where the seller faces a non-trivial problem of attracting the buyers. How should competitive bidding be organized in such a case?

Entering an auction often entails significant costs. In addition to the opportunity cost of time and effort spent on being physically present at an auction site, the preparation of bids may be costly as a result of concerns for due diligence. If bids are prepared at a stage where the eventual number of participants is unknown, it is natural to consider the performance

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of auctions where the number of bidders is random and endogenous. We call such trading mechanisms *informal auctions* to distinguish from traditional *formal auctions* where the set of bidders is known at the time of placing the bids. [1]

We show that uncertainty in the number of actual bidders leads to new types of bidding equilibria and to new revenue results in single object auctions with affiliated common values. In particular, we demonstrate that informal first-price auctions perform better than their formal counterparts when entry costs are significant. In an affiliated auction, different auction rules result in different expected payoffs to bidders with different information at the bid preparation stage. With endogenous entry decisions, the type distributions of entering bidders reflect these payoff differences. We show that the selection induced by informal auctions generates more social surplus than the selection resulting from formal auctions. When a large number of potential entrants choose entry endogenously, the surplus is collected by the seller. This revenue result runs against the spirit of the linkage principle according to which the seller normally benefits from revealing information to the bidders.[2]

Our model is particularly relevant for the analysis of procurement auctions where attracting sufficient participation is often problematic. In a recent paper, Kang and Miller (2021) document the scarcity of bidders for U.S. federal government procurement contracts, and they point out that many contracts have a single relevant bidder. In our model with common values, optimal bidding changes radically if a bidder realizes that she is the only participant in the auction. The informal auction format makes it harder for the bidders to extract the surplus from trade when no competing bidders are present.

The previous literature on auctions with costly participation has analyzed quite extensively formal auctions with a separate entry stage. In the second stage, all entering bidders are informed of the realized number of participants as if arriving at an auction house, and bid optimally in the ensuing auction.

In an informal auction, each bidder knows the equilibrium entry strategy of other potential participants but not the realized number of bidders at the time of choosing her bid. This auction format is meant to capture open selling procedures where interested bidders are invited to submit bids and the highest bidder wins and pays her own bid, as is typical in procurement. For independent private values, the usual mechanism design approach allows for the use of informal auctions, but the revenue equivalence theorem limits the significance

[1] We use this terminology for the ease of exposition only. Other authors, in particular DeMarzo, Kremer, and Skrzypacz (2005) uses 'informal auctions' to indicate lack of commitment to the auction mechanism

[2] The possible reversal of the linkage principle of Milgrom and Weber (1982) in informal auctions was observed by Matthews (1987) in an affiliated private values auction with the same exogenous distribution of bidders across the different auction formats.

of disclosing the realized number of bidders as a policy instrument.

We assume throughout that the true value of the good to the buyers is a binary random variable (the state variable of our model). We assume that trade is efficient in both states after accounting for the entry cost. Prior to any entry decisions, a large number of potential bidders have observed an affiliated private signal on the value of the object.

Our main objective is to analyze how different auction formats influence selection, i.e. the realized distribution of types amongst the participating bidders. Affiliation is crucial for such effects. With statistically independent signals, the revenue equivalence theorem implies that all efficient auctions result in equilibria where only the most optimistic bidders (i.e. bidders with the highest posterior on the high state) participate and the seller's expected revenue does not depend on the auction format.

Our first insight concerns efficient entry selection. We show that a utilitarian social planner restricted to symmetric entry strategies often induces entry by both the most optimistic and the most pessimistic bidders. With common values, the planner trades off duplicating the entry costs and missing out on profitable trades when no bidder enters. By affiliation, optimistic bidders are abundant if the value of the object is high, but they are relatively scarce if the value of the object is low. When the entry costs are relatively low, the optimal way to balance expected entry across the different states is by inducing entry from the most pessimistic bidders as well.

A qualitatively similar entry pattern occurs also in equilibrium. Affiliation implies that different bidders have different beliefs about the level of competition at the bidding stage. An individual bidder's signal has therefore two separate effects. The direct effect of a high signal is that the bidder is more optimistic about the value of the object. The indirect effect is that the other bidders are more likely to be optimistic as well. Hence a bidder with a high signal expects more aggressive bidding from other bidders, i.e. perceives the auction as more competitive.

These two effects are present in standard auctions as well, but entry costs highlight the second effect. Entry costs are sunk regardless of the outcome in the bidding stage. We show that this may lead to non-monotonic entry behavior in the sense that both optimistic and pessimistic bidders enter and get the same ex-ante expected payoff. We also show that uncertainty and in particular different beliefs about the number of opponents gives rise to non-monotonic bidding in the sense that participating bidders with high signal sometimes lose in informal auctions to bidders with lower signals.

Affiliated auctions with uncertain numbers of bidders are hard to analyze precisely because equilibria in monotone strategies may fail to exist.[3] Without monotonicity, it is not clear how one should proceed in the standard affiliated model with a continuum of signals as in Milgrom and Weber (1982). In fact, Lauermann and Speit (2022, forthcoming) show that with exogenous entry rates, equilibria may fail to exist altogether. In order to make progress, we assume that the signals are drawn from a finite set and they are independent across the potential bidders conditional on the true value of the object.

We model the auction as a Poisson game where the type of each potential player is affiliated with the value of the object. The number of entrants of each type is drawn from a Poisson distribution with a parameter that depends on the true binary value of the object and an endogenously determined entry rate for each type. An equilibrium of this game is a distribution of entrants and bids such that all entrants can cover their cost by their expected profit and no potential bidder of any type can make a strictly positive profit by entering.

One key feature of our equilibrium construction is that only extreme bidder types enter in equilibrium. We base our analysis on the bidders' expected payoffs conditional on the state. With a binary state and affiliated signals, the beliefs of the potential bidders on the state of the world are monotone in their signals. In equilibrium, the expected payoff at the auction stage must equal the entry cost for any type entering at a positive rate. If the expected payoffs in the two states differ at a fixed bid, only those potential bidders that assign the highest probability to the more favorable state make that bid. This implies that only bidders with the most extreme beliefs enter in equilibrium, and we may restrict attention to models with only two bidder types present at the bidding stage. We establish the existence and uniqueness of a symmetric equilibrium outcome in the overall game and we compute the equilibrium entry rates and equilibrium bids for our model.[4]

We get our strongest results when the entry cost is relatively high (so that the expected number of entrants is relatively low), and signals are strongly affiliated. In this case, placing a zero bid is in the support of the symmetric equilibrium bid distribution for both entering types. With atomless bidding strategies, this means that the expected payoff of each type equals the value of the good conditional on being the only entrant. This private benefit coincides with the planner's benefit from inducing additional entry when restricted to symmetric

[3] Landsberger (2007) is an early paper showing that existence of monotone equilibria often fails in auctions with participation costs.

[4] Neither uniqueness or existence is obvious from the outset in infomal auctions. Lauermann and Wolinsky (2017) discuss multiplicity of equilibria due to atoms in bidding distributions in their model of first-price auction with an unknown number of bidders. Lauermann and Speit (2022, forthcoming) show that a first-price auction with an exogenously given probability distribution for the number of participants may even fail to have an equilibrium.

strategies. This means that the symmetric equilibrium entry rates maximize utilitarian welfare in the class of symmetric entry strategies. Since we have a large number of potential bidders, their expected payoff net of entry costs must be zero and as a result the seller receives the maximal symmetric surplus as her expected revenue. For these parameter values, we see then that the informal auction is the revenue maximizing mechanism in the class of all symmetric mechanisms.

We also show that when entry in the informal auction differs from the planner's solution, a relatively simple change to the auction rules restores the efficiency. This requires a suitable combination of two separate instruments. By setting an entry fee and by allowing negative bids up to a given level, the seller can, in principle, induce efficient equilibrium entry rates and collect the entire expected social surplus in revenues.

In the main body of the paper, we normalize the seller's valuation of the product at zero so that negative bids can be interpreted as (positive) bids below the seller's valuation. In a procurement auction, this corresponds to allowing bids above the value of the service to the organizer of the auction. An important caveat is that when bids below the sellers valuation are possible, the seller must be able to commit ex-ante to accepting them in case no better offers are available.[5]

1.1 Related Literature

Lauermann and Wolinsky (2017) analyze a model where the seller invites bidders to an auction and Lauermann and Wolinsky (2016) analyze a model of sequential search for trading partners in a similar setting. Our model is based on a different premise on the costs of identifying potential trading partners. If such costs are prohibitive, the seller must rely on trading partners' self selection into the mechanism. This leads us to analyze voluntary endogenous entry into auctions in a setting that is otherwise quite close to Lauermann and Wolinsky (2017). Atakan and Ekmekci (2021) analyzes a related model, where the choice between competing auctions creates an opportunity cost of participating, but their focus is on information aggregation as the numbers of bidders and objects grow large.

Endogenous entry into auctions has been modeled in two separate frameworks. In the first, entry decisions are taken at an ex ante stage where all bidders are identical. Potential bidders learn their private information only upon paying the entry cost. Hence these models can be though of as games with endogenous information acquisition.[6] French and McCormick

[5] The seller would have an incentive to submit a shill bid equal to her valuation for the object

[6] The equilibrium determination of information accuracy in common values auctions started with Matthews (1977) and Matthews (1984). Persico (2000) extended this line of research to revenue comparisons

(1984) gives the first analysis of an auction with an entry fee in the IPV case. Harstad (1990) and Levin and Smith (1994) analyze the affiliated interdependent values case. These papers show that due to business stealing, entry is excessive relative to social optimum. They also show that formal second-price auctions dominate the formal first-price auction in terms of expected revenue. Informal auctions are thus not covered in these papers.

In the other strand, bidders decide on entry only after knowing their own signals. Samuleson (1985) and Stegeman (1996) are early papers in the independent private values setting where this question has been taken up. More recently, Bhattacharya, Roberts, and Sweeting (2014) and Sweeting and Bhattacharya (2015) analyze IPV models with selective entry where entry decisions are conditioned on private information on the true valuation.

Matthews (1987) analyzes a first-price auction with affiliated private values with an exogenously given random number of participants. He proves a reverse linkage principle result that implies higher revenues for the seller if the realized number of bidders is kept secret. In contrast to our paper, Matthews (1987) assumes monotonicity of best responses in signals and in the number of bidders (when disclosed), but does not characterize when equilibria in such strategies exist.[7] Pekec and Tsetlin (2008) provides an example of a common value auction where informal first-price auction results in a higher expected revenue than an informal second-price auction. The distribution of the bidders in that paper is somewhat extreme and not derived from entry decisions. Lauermann and Speit (2022, forthcoming) demonstrate the non-existence of equilibria in a common values first-price auction with an exogenously random number of bidders.[8]

Since we concentrate on symmetric equilibria of a game with a large number of potential entrants, our model has some similarities to the urn-ball models of matching. Similar to those models, our insistence on symmetric equilibria can be seen as a way of capturing a friction in the market that precludes coordinated asymmetric decisions. A recent example of such models is Kim and Kircher (2015) that studies matching with private values uncertainty. This approach has also been used in Jehiel and Lamy (2015) in a procurement auction setting with private (asymmetric) values. These models do not consider the case of common values.

The paper is structured as follows. Section 2 presents the main model. Section 3 goes through some preliminary results to set the stage for our main results. Section 4 characterizes

for different auction formats. Since the number of bidders and equilibrium information acquisition decisions are deterministic, equilibrium bidding in these papers is still as in standard affiliated auctions models.

[7] Pinkse and Tan (2005) have shown that the monotonicity in the number of bidders in the formal auction can be problematic.

[8] In contrast to our paper, their analysis relies on a modeling assumption guaranteeing that best replies are monotonic in type and they do not consider revenue properties of different auction formats.

the unique equilibrium in both informal and formal auction models. Section 5 compares the expected revenues across different auction formats and Section 6 discusses the modeling assumptions in this paper. All proofs omitted from the main text are in the Appendix.

2 Model

A seller has a single indivisible object for sale. The value of the object, $v(\omega)$, is common to all potential bidders and determined by a binary random variable $\tilde{\omega}$ with realizations $\omega \in \{0,1\}$. There is a pool of potential bidders who choose whether or not to enter the auction. The cost of entering and making a bid is $c > 0$ for an individual bidder.[9] We assume that $v(1) - c > v(0) - c > 0$ and normalize the value to the seller at zero. This implies that $v(\omega) - c$ is the social surplus from trading in state ω.

Each potential bidder observes a signal $\widetilde{\theta}$ with realizations in $\{\theta_1, ..., \theta_M\}$. The common conditional distribution of signals in state $\omega \in \{0,1\}$ is denoted by:

$$\alpha_{\omega,m} =: \Pr\{\widetilde{\theta} = \theta_m \,|\widetilde{\omega} = \omega\} \text{ for } m \in \{1, ..., M\}. \tag{1}$$

We label the signals so that they satisfy strict monotone likelihood ratio property: $\frac{\alpha_{1,m}}{\alpha_{0,m}}$ is increasing in m.

We model the number of bidders that participate as a Poisson random variable with an *endogenously* given parameter.[10] Denote the endogenous entry rate of type m entrants by π_m. Given the affiliated signal structure, the realized number of participants of type m is then given by a Poisson random variable N_ω^m with parameter $\lambda_{\omega,m} := \alpha_{\omega,m}\pi_m$ that depends on π_m and the exogenously given signal distribution (1) across the two states. This model is the natural limit as $N \to \infty$ of a game with N potential bidders who get private signals according to (1) and play a symmetric mixed strategy in entry probabilities.[11]

[9] We comment on possible interpretations of this common participation cost at the end of this section.

[10] For a discussion of the Poisson entry model in the context of a procurement auction with private values, see Jehiel and Lamy (2015).

[11] To see how one arrives naturally at the Poisson model, assume that there are N potential entrants who observe signals according to (1). If each entrant with signal m enters with probability $\pi_{m,N}$, then the number of type m entrants is a binomial random variable with parameters $(\alpha_{1,m}\pi_{m,N}, N)$ and $(\alpha_{0,m}\pi_{m,N}, N)$ in states $\omega = 1$ and $\omega = 0$, respectively. When $N \to \infty$ and at the same time $\pi_{m,N}$ decreases so that $N\pi_{m,N}$ converges (this must happen, because bidding is voluntary and hence expected payoff of participation must be weakly positive in equilibrium), the number of entrants of type m in state ω approaches a Poisson distribution with a parameter proportional to $\alpha_{\omega,m}$. In earlier working paper versions of this paper, we have shown that the main results of our model hold when there are only two potential participants. Details are available upon request.

In the *informal auction*, participating bidders place their bids before learning the realized number of other participating bidders. A bidder with the highest bid wins and pays her own bid.[12] A symmetric strategy profile is a pair (b, π) where $b : \{1, ..., M\} \to \Delta(\mathbb{R}_+)$ is a distribution on positive bids for each type of participating bidders and $\pi : \{1, ..., M\} \to \mathbb{R}_+$ is a vector of entry rates. By a symmetric equilibrium of the informal auction, we mean a pair (b, π), such that no bid gives an expected payoff strictly above c for any bidder type. If $\pi_m > 0$, then the expected payoff from all bids in the support of the bid distribution of type m is exactly c.

We will later compare the informal auction with a *formal* (first-price) auction where the bidders learn the number of competing bidders before placing their bids. In the formal auction, the entry strategy is as above, but the bidding strategy $b : \{1, ..., M\} \times \mathbb{N} \to \Delta(\mathbb{R}_+)$ conditions on the realized number of participating bidders. As with the informal auction, a bidder with the highest bid wins and pays her own bid.

Remark 1 *The common participation cost may also reflect the opportunity cost of participating in other similar auctions. This cost is independent of own signal in the current auction if the values of the goods across the auctions are independent and if the signals of the potential bidders are i.i.d. conditional on the true value of the good. This interpretation applies in particular to capacity constrained bidders in a procurement auction.*

3 Preliminary Analysis of the Informal Auction

In this section, we derive a number of preliminary results. In Subsection 3.1, we derive the optimal entry profile for a social planner restricted to anonymous strategies. We show that the optimum in this benchmark can be obtained by profiles where only bidders with signals θ_1 or θ_M enter with positive probability. In Subsection 3.2, we show that the equilibrium bidding strategies are atomless. In Subsection 3.3, we show that also equilibrium entry profiles feature only the two types θ_1 and θ_M. Finally, in Subsection 3.4, we fix an exogenous entry profile by the extreme types and derive the corresponding atomless bidding equilibrium. We use these results in the next section to characterize the full equilibrium with endogenous entry.

3.1 Symmetric Planner's Optimum

We start by analyzing the symmetric benchmark where the social planner chooses entry rates $\pi = (\pi_1, ..., \pi_M)$ to maximize the expected gain from allocating the object net of the

[12] We show below that ties cannot happen with positive probability in informal auctions.

expected entry costs. The expected total surplus $W(\pi)$ depends only on the total expected number of entrants $\Sigma_m \alpha_{\omega,m}\pi_m$ in each state ω:

$$W(\pi) = q[v(1)\left(1 - e^{-\Sigma_m \alpha_{1,m}\pi_m}\right) - c\Sigma_m \alpha_{1,m}\pi_m]$$
$$+ (1-q)\left[v(0)\left(1 - e^{-\Sigma_m \alpha_{0,m}\pi_m}\right) - c\Sigma_m \alpha_{0,m}\pi_m\right].$$

The planner's problem is to find $\max_{\pi \geq 0} W(\pi)$. The first order condition with respect to the entry rate π_m of type m can be written as:

$$q_m V^1(\pi) + (1 - q_m) V^0(\pi) = c, \tag{2}$$

where $V^\omega(\pi) = v(\omega)e^{-\Sigma_m \alpha_{\omega,m}\pi_m}$ is the gain in social surplus in state ω from increasing entry rate π_m and where q_m is the posterior of state $\omega = 1$ conditional on signal m:

$$q_m = \frac{\alpha_{1,m} q}{\alpha_{1,m} q + \alpha_{0,m}(1-q)}.$$

This condition must hold for any m for which entry rate π_m is strictly positive. Since the posteriors q_m are strictly increasing in m by affiliation, we see that entry by multiple types is possible only if entry is optimal state by state, i.e.:

$$V^\omega(\pi) = c \text{ for } \omega \in 0, 1.$$

Proposition 1 below shows that such a solution exists whenever the entry cost c is low enough or signals are strongly affiliated in the sense that $\frac{\alpha_{1,M}}{\alpha_{0,M}}$ is large enough. Furthermore, an optimal solution can be obtained at an extremal entry rate vector π such that $\pi_m = 0$ for $1 < m < M$.

Proposition 1 *The symmetric planner's problem has a solution $\pi^* = (\pi_1^*, 0, ..., 0, \pi_M^*)$ with $\pi_M^* > \pi_1^* \geq 0$. At this solution, $\pi_1^* > 0$ if and only if:*

$$\alpha_{0,M} \log\left(\frac{v(1)}{c}\right) < \alpha_{1,M} \log\left(\frac{v(0)}{c}\right),$$

or equivalently,

$$c < \bar{c} := e^{\frac{\alpha_{1,M}\log(v(0)) - \alpha_{0,M}\log(v(1))}{\alpha_{1,M} - \alpha_{0,M}}}.$$

3.2 No Pooling with Endogenous Entry

In any entry equilibrium, entrants earn an expected profit of c at the bidding stage. Hence for all bids p in the union of the bid supports of all bidders, the equilibrium payoff of any bidder submitting p must be c, and the payoff for any other type cannot exceed c since otherwise we would have a profitable deviation for some individual potential entrant. In this section, we use this property to rule out the possibility of pooling bids (i.e. bids chosen with a strictly positive probability by some bidder type).

With a random number of bidders, the analysis of pooling bids is more delicate than in the case of a fixed number of bidders. With uniform tie-breaking for highest bids, a bidder submitting a pooling bid is more likely to win if the number of tying bids is small. The number of tying bids contains information on the realized number of participating bidders of different types. Since the distribution of different types depends on state, such information is in turn informative on the state of the world and hence on the value of the object. The additional information contained in the event of winning the auction must be accounted for when calculating the optimal bid. Lauermann and Wolinsky (2017) show that such rationing effects may indeed lead to bidding equilibria with pooling.

We now argue that this cannot occur in our model with endogenous entry. As observed by Pesendorfer and Swinkels (1997), it is not possible to have pooling bids that are made only by the highest types. If they were the only bidders to pool on a bid p, then the rationing effect would be negative: winning at bid p is more likely when there are few tied bidders. But if only high type bidders bid p, then winning with a tied bid decreases the posterior on $\{\omega = 1\}$ relative to the value of the object conditional on the event that the bidder is tied for the highest bid. By bidding $p + \varepsilon$, for a small enough ε, the bidder wins in the event of a tied bid without any rationing and makes a positive gain from the deviation. As a result, pooling by high types only is not possible in equilibrium.

Denote by $R_\omega\left(p\right)$ the expected equilbrium rent at the auction stage in state ω from bid p:

$$R_\omega\left(p\right) = \left(v\left(\omega\right) - p\right) \prod_{m=1}^{M} e^{-\lambda_{\omega,m}\left(1-F_m(p)\right)},$$

where $F_m\left(p\right)$ is c.d.f. of the bid distribution of type θ_m. Since this rent is conditional on the state, bidders of all types agree on $R_\omega(p)$. To rule out other types of pooling bids, we note first that there cannot be pooling at any $p < v\left(0\right)$. This is because for such a low bid, $R_\omega(p) > 0$ for $\omega \in \{0, 1\}$ and a deviation to a slightly higher price would be profitable.

For $v(0) \leq p < v(1)$, $R_0(p) \leq 0 < R_1(p)$. By affiliation, $\Pr\{\omega = 1\,|\theta_m\}$ is increasing in m. Therefore, if type θ_m gets an expected payoff of c by bidding $p \geq v\left(0\right)$, then type θ_M gets strictly more than c at the same bid. This is not compatible with interim entry equilibrium. Since it is not possible to have pooling bids that are made only by high types, pooling is not possible for any $p \geq v\left(0\right)$ either. We have hence proved:

Lemma 1 *There are no atoms in the bidding distribution b of a symmetric equilibrium (b, π).*

3.3 Entry by Extreme Types

We consider next the types of bidders that can enter in an equilibrium with atomless bidding strategies. For each p, there are three possibilities: either i) $R_1(p) = R_0(p)$, ii) $R_1(p) > R_0(p)$, or iii) $R_1(p) < R_0(p)$. All types of bidders are indifferent between entering and submitting a bid p in case i). In case ii), only type θ_M can make bid p in equilibrium with interior entry probabilities. If θ_m, $m < M$, makes the bid, her expected payoff at the bidding stage is $R_0(p) + q_m(R_1(p) - R_0(p))$ and she must earn at least c to be willing to enter and bid p. But since $q_m < q_M$, type θ_M will get strictly more than c by entering and bidding p, which cannot be an equilibrium. Similarly in case iii), only type θ_1 can enter in equilibrium.

Our equilibrium construction shows that all the cases i) - iii) are possible, but the bid distribution is fully pinned down by the parameters of the bidders that bid in cases ii) and iii). The only remaining indeterminacy concerns the exact composition of bidders that make bids where case i) holds. But the requirement that the expected payoffs be equalized across the two states determines the aggregate distributions of bids for each state in a manner analogous to the payoff equalization across the two states in the planner's problem. As in the planner's case, these aggregate distributions can be generated with positive entry by types θ_1 and θ_M. We formalize this discussion in the following Lemma:

Lemma 2 *Let $\left(\bar{b}, (\bar{\pi}_1, \bar{\pi}_M)\right)$ denote an equilibrium of a reduced model, where only types $\{\theta_1, \theta_M\}$ exist. Then $\left(\bar{b}, (\bar{\pi}_1, 0, ..., 0, \bar{\pi}_M)\right)$ remains an equilibrium of the full model that allows entry by all types $\{\theta_1, ..., \theta_M\}$. Moreover, the equilibrium distribution of bids (and therefore also the expected revenue) in any other equilibrium (b, π) of the full model coincides with the equilibrium distribution of bids in an equilibrium of the reduced model.*

3.4 Bidding Equilibrium of the Two-type Model

Based on Lemmas 1 and 2, we can restrict our attention to atomless bidding equilibria of a model with binary bidder types $\theta \in \{l, h\}$, where labels l and h correspond to the extreme types θ_1 and θ_M of the full model, respectively. In this subsection, we characterize further the set of possible bidding equilibria b. The full equilibrium with endogenous entry will be analyzed in the next section.

Recall that the number of bidders with signal θ in state ω is a Poisson random variable N_ω^θ with parameter $\lambda_{\omega,\theta} = \alpha_{\omega,\theta} \cdot \pi_\theta$. With affiliated signals, $\alpha_{1,h} > \alpha_{0,h}$ and $\alpha_{1,l} < \alpha_{0,l}$.

This in turn implies that $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{1,l} < \lambda_{0,l}$ for any entry profile $\pi = (\pi_h, \pi_l)$. The expected payoff from a bid p to a bidder of type θ when the other players bid according to the profile $\mathbf{F} = (F_h(\cdot), F_l(\cdot))$ is:

$$\begin{aligned} U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))} e^{-\lambda_{1,l}(1-F_l(p))} (v(1) - p) \\ &\quad + (1-q_\theta) e^{-\lambda_{0,h}(1-F_h(p))} e^{-\lambda_{0,l}(1-F_l(p))} (v(0) - p). \end{aligned}$$

A symmetric bidding equilibrium for entry rate profile π is a bid profile $\mathbf{F}$ such that for all $p \in \mathrm{supp} F_\theta(\cdot)$, p maximizes $U_\theta(p)$. Using this property and the functional form of $U_\theta(p)$, we can pin down some key properties of a bidding equilibrium. Lemma 3 below shows that depending on the entry rate profile, we are in one of two qualitatively distinct cases. The first case is just a discrete type version of a standard monotonic equilibrium: the higher type always bids higher than the low type. The second case is novel to our model and shows that the bidding equilibrium may fail to be monotonic. The bid supports overlap and hence a low type wins against a high type with a positive probability.

Lemma 3 *Let b be an atomless bidding equilibrium for given entry rate profile $\pi = (\pi_h, \pi_l)$. Then the union of the bid supports of the two types is an interval and the bid support of the low type is an interval that contains 0. If*

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} < \frac{v(1)}{v(0)},$$

then the bid supports are non-overlapping intervals with a single point in common. If

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} > \frac{v(1)}{v(0)},$$

then the bid supports intersect and 0 is contained in the supports of both types.

Bid supports overlap if the value of the object does not differ too much across the two states or if the signals are sufficiently correlated. In the extreme case, where the signals are perfectly correlated, the two extreme types randomize on an interval starting at $p = 0$. Lemma 3 establishes a threshold value for $\frac{v(1)}{v(0)}$ to maintain monotone equilibria for a given level of correlation.

It should be emphasized that we have not yet proven the existence of an equilibrium in our model. The Lemmas above should therefore be understood as a characterization of a candidate for an equilibrium bidding strategy. Lauermann and Speit (2022, forthcoming) show that a bidding equilibrium may fail to exist in a first-price auction with an exogenously given type distribution. This possibility is present in our model with exogenous entry as well.[13] In the next section we show that with endogenous entry, our model has always a unique symmetric bidding equilibrium and so the characterization in Lemma 3 is valid.

4 Equilibrium with Endogenous Entry

In this section, we consider the equilibrium entry rates in informal and formal auctions. We analyze their properties through a comparison with the symmetric social planner's solution. Therefore, we set the stage in subsection 4.1 by explaining the distinction between private and social entry incentives. In subsection 4.2, we analyze the equilibrium of the informal auction and in subsection 4.3, we analyze the equilibrium of the formal auction.

4.1 Private and Social Incentives for Entry

It is helpful to start by considering a hypothetical game where the object is given for free to the entrant if there is a single entrant. With two or more entrants the object is withdrawn from the market but the entrants have to pay the entry cost. Denote by $V_\theta^0(\pi_h, \pi_l)$ the expected payoff to a bidder of type $\theta \in \{h, l\}$ at the auction stage with the symmetric entry profile (π_h, π_l):

$$\begin{aligned} V_h^0(\pi_h, \pi_l) &= q_h e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l} v(1) + (1 - q_h) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l} v(0), \\ V_l^0(\pi_h, \pi_l) &= q_l e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l} v(1) + (1 - q_l) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l} v(0). \end{aligned}$$

Any (π_h, π_l) such that $V_h^0(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l) = c$ in the above equations guarantees that every potential entrant is indifferent between entering and staying out. By inspection, we see that this is the same condition as in equation (2) characterizing the interior planner's solution. This implies that private incentives for entry coincide with social incentives. To understand why this is the case, note that in this hypothetical situation each entrant gets exactly her marginal contribution to the social welfare as her payoff. Since the value of the object is common to all the players, an entrant contributes to the total surplus if and only if she is the only entrant. Since the object is given for free to the sole entrant, her private benefit equals her marginal contribution and the private value of entry coincides with the social value of entry. We show below that our different auction formats give (at least weakly) too high entry incentives for the high types. Quantifying this excessive entry incentive across different auction formats gives us our revenue comparisons.

[13] We thank Andrea Speit for pointing this out to us.

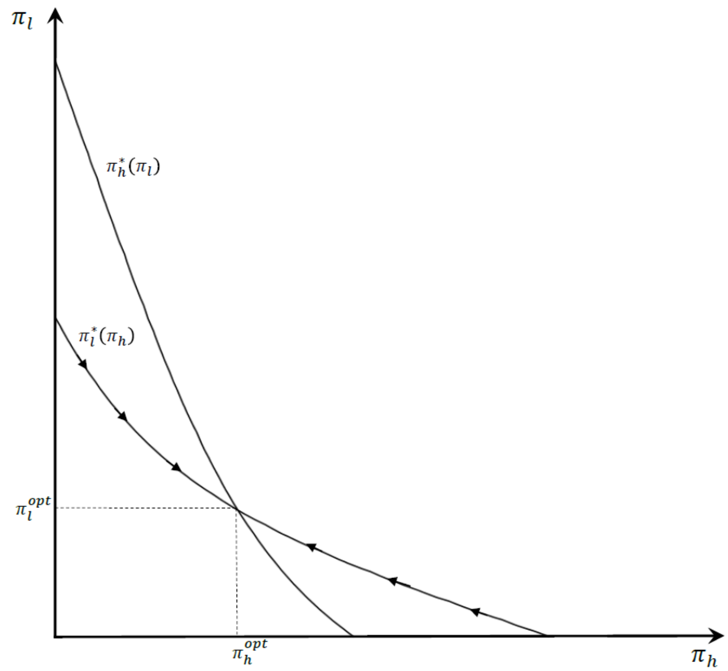

Figure 1: Social planner's reaction curves.

In Figure 1, we illustrate the social optimum in the case with interior entry probabilities by drawing in the (π_h, π_l) −plane the planner's reaction curves:

$$\begin{aligned}\pi_l^*(\pi_h) \quad &: \quad = \arg\max_{\pi_l \geq 0} W(\pi_h, \pi_l), \\ \pi_h^*(\pi_l) \quad &: \quad = \arg\max_{\pi_h \geq 0} W(\pi_h, \pi_l),\end{aligned}$$

where $W(\pi_h, \pi_l)$ is the expected total surplus at entry profile (π_h, π_l):

$$\begin{aligned}&W(\pi_h, \pi_l) = q[v(1)\left(1 - e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l}\right) - c(\alpha_{1,h}\pi_h + \alpha_{1,l}\pi_l)] \\ &+ (1-q)\left[v(0)\left(1 - e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l}\right) - c(\alpha_{0,h}\pi_h + \alpha_{0,l}\pi_l)\right].\end{aligned}$$

The social optimum $\left(\pi_h^{opt}, \pi_l^{opt}\right)$ is at the intersection of these curves. These curves are also the indifference contours $V_h^0(\pi_h, \pi_l) = c$ and $V_l^0(\pi_h, \pi_l) = c$ for a potential entrant of type h and l, respectively, who gets her marginal contribution as expected payoff.

Auctions often give an additional reward to high type entrants on top of their social contribution. This distorts equilibrium entry rates from socially optimal levels. Since bidding zero is in the support of the low type bidders by Lemma 3, their equilibrium entry incentives coincide with the planner's incentives conditional on the entry rate of the high types.[14] As

[14]This is true also in the formal auction, where the low type gets rewarded if and only if no other bidder

a result, their equilibrium reaction curve coincides with the planner's reaction curve and hence any equilibrium entry profile must be on the planner's reaction curve $\pi_l^*(\pi_h)$ for the low type. For our revenue comparisons, it is therefore useful to analyze how movements along that curve change the total surplus.

We denote by $W^*(\pi_h)$ the total surplus when low types react optimally to π_h:

$$W^*(\pi_h) := W(\pi_h, \pi_l^*(\pi_h)).$$

The following Lemma shows that $W^*(\pi_h)$ is single peaked in π_h with its peak at π_h^{opt}. This is illustrated in Figure 1 by the arrows along $\pi_l^*(\pi_h)$ that point towards increasing social surplus. Based on this Lemma we can compare auction formats simply by analyzing how far they distort the entry rate of the high type away from the social optimum.

Lemma 4 *Let $\pi_h^{opt} > 0$ denote the socially optimal high type entry rate. We have*

$$\frac{dW^*(\pi_h)}{d\pi_h}\begin{cases} > 0 \text{ for } \pi_h < \pi_h^{opt} \\ = 0 \text{ for } \pi_h = \pi_h^{opt} \\ < 0 \text{ for } \pi_h > \pi_h^{opt} \end{cases}. \tag{3}$$

Notice that by Proposition 1 the planner's solution has $\pi_l^{opt} > 0$ only when $c < \bar{c}$. If $c \geq \bar{c}$, there is no entry by the low type in the planner's solution and this must be the case also in equilibrium, irrespective of whether the auction is formal or informal.[15] Consequently, as all the entering bidders have observed a high signal, the expected payoff to the entering bidders is given by the probability that no other bidder entered times the expected value of the object conditional on that event.[16] Since the updated belief of a high type bidder on $\{\omega = 1\}$ is given by $q_h = \frac{q\alpha}{q\alpha+(1-q)\beta}$, equating the expected benefit from entry to the cost of entry gives:

$$q_h e^{-\alpha\pi_h} v(1) + (1-q_h)\, e^{-\beta\pi_h} v(0) = c,$$

which coincides with the planner's optimality condition. Hence, the symmetric equilibrium entry profile in both auction formats coincides with the planner's optimal solution. Since the bidders expected payoff is zero, the seller collects the entire expected social surplus in expected revenues and hence both auction formats are also revenue maximizing within the

is present. We characterize the bidding equilibrium for that case in Lemma 6 below.

[15]The low type entry incentives are always aligned with the social planner: Lemmas 3 and 6 below guarantee that low types cannot get more than their social contribution in either auction format.

[16]With only one entering type, the bidding equilibrium in an informal auction is a connected interval containing zero. In the formal auction, a bidder bids zero if no other bidder is present and otherwise bids the expected value of the object conditional on the number of entering bidders.

class of symmetric mechanisms (where we require symmetry at the entry stage as well as at the bidding stage):

Remark 2 *If $c \geq \bar{c}$, then only the high type enters in the planner's solution and both auction formats are efficient and hence revenue equivalent.*

4.2 Entry Equilibrium in Informal Auctions

We now consider the entry equilibrium in the more interesting case $0 < c < \bar{c}$, where the planner's solution induces entry by both types. We use the three Lemmas of the previous section to narrow down the set of possible equilibria in various ways. By Lemma 1 there are no symmetric equilibria with pooling bids, so we can concentrate on atomless bidding strategies. By Lemma 2 we can concentrate on the lowest and highest type of potential entrants. Finally, Lemma 3 further narrows down the form of bidding strategies to two distinct cases. In this section, we show the existence of a unique equilibrium consistent with these Lemmas.

We denote by $V_\theta^{IF}(\pi_h, \pi_l)$ the expected payoff of a bidder with type θ at the bidding stage of the informal first-price auction for given entry rates π_h, π_l. We show that there is a unique pair $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ such that $V_h^{IF}\left(\pi_h^{IF}, \pi_l^{IF}\right) = V_l^{IF}\left(\pi_h^{IF}, \pi_l^{IF}\right) = c$.

In addition to the uniqueness, the proposition below contains a qualitative statement about the equilibrium. Each low type entrant gets a payoff in the bidding stage that is exactly equal to her social contribution:

$$V_l^{IF}(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l),$$

which means that equilibrium entry profile $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ must be located along the planner's reaction curve $\pi_l^{IF} = \pi_l^*\left(\pi_h^{IF}\right)$. High type bidders may get a higher payoff. In order to dissipate the excess rent of the high types, we must have $\pi_h^{IF} \geq \pi_h^*\left(\pi_l^{IF}\right)$. As a result, the equilibrium entry rate must be (weakly) too high for the high type and (weakly) too low for the low type in comparison to the planner's optimum.

Proposition 2 *Let $0 < c < \bar{c}$. The informal first-price auction has a unique symmetric equilibrium where only the extreme types enter with strictly positive rate. The entry rates by the highest and lowest types, respectively, satisfy $\pi_h^{IF} \geq \pi_h^{opt}$ and $\pi_l^{IF} = \pi_l^*\left(\pi_h^{IF}\right) \leq \pi_l^{opt}$. All entering bidder types use atomless bidding strategies. $p = 0$ is always in the bid support of the low types, and the the upper bound of the bid support of high types is given by:*

$$p^{\max} := q_h v(1) + (1 - q_h) v(0) - c.$$

Any equilibrium (b, π) with positive entry by interior types induces the same probability distribution of bids and revenues.

4.3 Entry Equilibrium in Formal Auctions

We showed in Lemma 2 above that for informal auctions, we can restrict our analysis to the two-type model. The same is true for formal auctions. In fact, we can show that the bidding equilibria of formal auctions are simpler than in the informal auction in the sense that for all bid levels, either the lowest or the highest type has a strict advantage over any intermediate type. This implies that the intermediate types can never break even in equilibrium and so there cannot be any equilibria where intermediate types enter with positive rate.

Lemma 5 *Let $\pi = (\pi_1, ..., \pi_M)$ denote the vector of entry rates by different types in an equilibrium of the formal auction. Then $\pi_2 = ... = \pi_{M-1} = 0$, i.e. only types θ_1 and θ_M may enter with a strictly positive rate.*

Since only two types enter in equilibrium, we can use the following characterization of the unique symmetric bidding equilibrium in formal auctions taken from Chi, Murto, and Välimäki (2019).

Lemma 6 *With two bidder types and exogenous entry rates, the formal first-price auction format has a unique symmetric equilibrium. If $n = 1$, the only participant bids zero. With $n \geq 2$ participants, the low type bidders pool at bid*

$$\underline{p}(n) = E\left[v \,\middle|\, \theta = l,\ N^l = n-1,\ N^h = 0\right]$$

and the high types have an atomless bidding support $\left[\underline{p}(n), \overline{p}(n)\right]$ with some $\overline{p}(n) > \underline{p}(n)$.

High type bidders earn a higher private benefit in the bidding stage than their contribution to the social surplus. To see this, recall that in a model with common values additional entry is socially valuable only if no other bidder participates in the auction. In the formal auction high type bidders receive the social benefit (i.e. object for free if $n = 1$), but she also an extra information rent when bidding against low type bidders when $n \geq 2$. By bidding $\underline{p}(n) + \varepsilon$, a high type wins against low types, and her payment $\underline{p}(n) + \varepsilon$ is lower than her expectation of the value of the object given her signal. As a result, equilibrium entry is socially excessive in formal auctions.

Given the bidding equilibrium characterized above we can prove the existence and uniqueness of symmetric equilibria for formal auctions with endogenous entry. Proposition 3 below mirrors Proposition 2 for informal auctions. The main qualitative difference is that in formal auctions, equilibrium entry rates are always distorted away from the socially optimal levels as long as entry cost is low enough to induce entry by both types.

Proposition 3 *Let $0 < c < \overline{c}$. The formal first-price auction has a unique symmetric equilibrium where only the extreme types enter with strictly positive rate. The equilibrium entry rates by the highest and lowest types, respectively, satisfy $\pi_h^F > \pi_h^{opt}$ and $\pi_l^F = \pi_l^* \left(\pi_h^F\right) < \pi_l^{opt}$.*

Note that if $c \geq \overline{c}$, then the low type does not enter and Remark 2 applies.

5 Revenue Comparisons

5.1 Formal versus informal auctions

We compare next the expected equilibrium revenues of informal and formal auctions when multiple types of bidders enter in equilibrium, i.e. when $c < \overline{c}$. Our first result shows that when the affiliation is strong and the entry cost is sufficiently large, the informal auction gives the entire (symmetric) social surplus to the seller in expected revenues. To see when this happens, recall from Lemma 3 that the bid supports of both type of bidders must contain 0 whenever

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} > \frac{v(1)}{v(0)}.$$

The left hand side is decreasing in π_l, and approaches $\alpha_{0,l}/\alpha_{1,l}$ when $\pi_l \to 0$. This ratio measures the difference across the states of the probability of observing a low signal, and reflects the divergence in the beliefs of different bidder types on the degree of competition at the bidding stage. Whenever this ratio is larger than the ratio $v(1)/v(0)$ of payoffs across the states, a sufficiently high entry cost will result in a low entry rate for the low type bidders. In this case, a zero bid must be in the high-type bidders' support and they get no rent on top of their contribution to the social welfare in the informal auction. As a result, the informal first price auction maximizes the seller's expected revenue in the class of *all* symmetric mechanisms.

In contrast, in formal auctions the high-type will always get an information rent that strictly exceeds her social contribution and entry rates are distorted away from social optimum. This establishes the strict superiority of the informal first price auction.

Proposition 4 *If*

$$\frac{\alpha_{0,l}}{\alpha_{1,l}} > \frac{v(1)}{v(0)}, \tag{4}$$

then there is a $\widetilde{c} < \overline{c}$ *such that for* $c \in (\widetilde{c}, \overline{c})$*, entry is efficient in the informal auction and the expected revenue is strictly higher than in the formal auction.*

When (4) does not hold and/or entry cost is very low, the revenue comparison is less straightforward. In this case, the informal auction also induces too much entry by the high types and the distortions across the two formats must be compared. We show below that as long as the entry cost is substantial enough, the superiority of the informal auction continues to hold, but a more elaborate argument is needed. The key to understanding this is the linkage principle with uncertainty in the number of competing bidders.

The linkage principle by Milgrom and Weber (1982) says that the expected revenue increases if an auction format induces a positive linkage between a bidder's own signal and her expected payment, conditional on winning the auction. As explained by Milgrom and Weber (1982), one implication of this is that expected price increases if public information reinforces the linkage between bidders' signals and their expected payments. However, Matthews (1987) observes that in an auction with affiliated private values and exogenous random participation, revealing the number of bidders creates a negative linkage between a bidder's own signal and her expected payment. Therefore revealing the number of bidders decreases the seller's expected revenue. Both Milgrom and Weber (1982) and Matthews (1987) assume that the type distribution of the bidders is independent of the auction format. We discuss below how the insights from the linkage principle extend to our setting with endogenous participation.

In the informal auction, one's own signal has no effect on expected payment conditional on winning at the submitted bid. In the formal auction, in contrast, one's own signal influences the expected payment through its effect on the distribution of the number of bidders n conditional on winning. In a formal auction, n is revealed before the bidding stage and the equilibrium bid depends on n. Recall from Lemma 6 that for any realized n, low types pool at $\underline{p}(n)$ and the high types mix over interval $\left[\underline{p}(n), \overline{p}(n)\right]$. To isolate the linkage of own signal to expected payment in a model with discrete types and mixed strategies requires some care. Consider a bidding strategy where the bidder wins against low types but loses against all high types, i.e., bid $\underline{p}(n) + \varepsilon$ with ε vanishingly small. Since this bidding strategy is essentially optimal for both the low and high type, we can use it to isolate the linkage between own signal and expected payment.[17] The expected payment

[17] Note that this strategy is almost a best-response to both types of bidders and therefore allows meaningful comparison between expected payments across the signals even though it is not within equilibrium support

conditional on winning with that strategy is just the expectation of $\underline{p}(n)$ over n conditional on no high types entering, where we define $\underline{p}(1) \equiv 0$ since a solitary bidder bids zero.

By affiliation, a high type expects a smaller number of low type bidders and hence a lower n conditional on no high types. If $\underline{p}(n)$ were decreasing in n, the linkage principle would work in the normal way: a high type bidder expects to pay more than the low type. By affiliation, $\underline{p}(n)$ is indeed decreasing in n for $n \geq 2$ since the posterior on $\omega = 1$ is decreasing in the number of low type bidders. Notice, however, that $\underline{p}(1) = 0 < \underline{p}(2)$. This breaks the monotonicity of $\underline{p}(n)$. A high type bidder evaluates the probability that there are no low type bidders present to be higher than a low type bidder. This reduces the expected payment of the high type relative to the low type. If the entry rate of the low type bidders is small enough, this prospect of facing no competition dominates and makes the linkage between signal and expected payment negative.[18] If the entry rate of the low type is very high, we are no longer able to say which linkage effect dominates.

We conclude that whenever the entry rate of the low type bidders is sufficiently low, revealing the number of bidders introduces a negative linkage between a bidder's signal and her expected payment. If entry rates are fixed, this means that the high type bidder's expected rent is higher in the formal auction than in the informal auction. But when entry rates are endogenous, this further implies that the entry incentive of the high-type is distorted more in the formal than in the informal auction. Translated into the exogenous parameters of the model, this implies that whenever entry cost is high enough, the informal first-price auction dominates formal first-price auction in terms of expected revenue.

Proposition 5 *There is a $c'' < \bar{c}$, such that for $c \in (c'', \bar{c})$, informal first-price auction generates a strictly higher expected revenue than formal auctions.*

5.2 Entry Fees and Reserve Prices

One can ask if the seller can increase her expected revenue by other auction design features such as entry fees or reserve prices. If the conditions in Proposition 4 hold, then the informal auction induces the highest possible total surplus and gives this entirely to the seller. In that case it is clear that no symmetric mechanism can improve the expected revenue.

If the informal auction is inefficient, then there is a distortion in entry rates that could potentially be reduced or even eliminated. The distortion results from the positive equilib-

for the low type bidders. In the limit as $\varepsilon \to 0$, types are indifferent between bidding $\underline{p}(n)$ and $\underline{p}(n) + \varepsilon$. The only effect of deviating up by ε is that this will tilt the rationing to the deviator in case of a tie at $\underline{p}(n)$.

[18] In a model with two potential bidders, this is always the case.

rium rent of the high-type bidders. Hence it would be desirable to find an instrument that penalizes high types but not the low types.

An entry fee is an instrument that penalizes both types equally. As a result, adding a small entry cost modifies the entry incentive of the high types in the intended direction, but at the same time affects the low types in the wrong direction. An argument based on the envelope theorem suggests that a small entry cost could marginally increase revenue. To see this, note that in equilibrium, entry by the low types is socially optimal conditional on the entry rate of the high types. Therefore, a slight change in the low type entry rate will not have a first-order effect on the total surplus. In contrast, the high type entry rate is already distorted away from social optimum and therefore a small change has a first-order effect in total surplus and revenue.

To improve entry selection further one would need to compensate for the penalizing effect of the entry fee to the low types with another instrument. In the main part of our paper, we have set the minimum bid equal to the value $v = 0$ of the object to the seller. Suppose that the seller can commit to accepting bids below v (i.e. negative bids in the main analysis). Since lowering the minimum bid will be particularly beneficial to a bidder that happens to be alone in the market, and since such a scenario is more likely from the point of view of a low type bidder, this instrument indeed rewards more the low types than the high types. By suitably combining these two instruments, an entry fee and a reduced minimum bid, we can indeed modify the entry incentives of both types of bidders to achieve socially optimal entry selection. We show in the Proposition below that this works both in the context of informal and formal auctions.

Notice that in the procurement context, lowering the minimum bid below the seller's value should be interpreted as increasing the maximum bid above the auctioneer's value for the service. We should point out that committing to accepting bids below v (or above v in the procurement context) may be very hard in practice. If the seller can make shill bids, she has always an incentive to bid v at the auction stage. If the potential participants anticipate this, entry rates will not respond as intended. In order to manipulate the equilibrium entry rates, the auctioneer must find a way to credibly commit not to make shill bids.

Proposition 6 *If the equilibrium entry profile of an informal or a formal auction is inefficient, then a suitably chosen combination of a positive entry fee $f > 0$ and a negative minimum bid $-\underline{b} < 0$ yields an equilibrium with socially optimal entry profile $\left(\pi_h^{opt}, \pi_l^{opt}\right)$.*

6 Discussion

6.1 Auctions vs. Procurement

Our model is written as an auction where the seller has a known valuation of zero for the object that she is selling. This means that the values $v(0)$ and $v(1)$ should be interpreted as gains from trade in the two states. The lower bound for bids is then set at the value of the object to the seller.

The model can also be interpreted as a reverse auction as typically used in procurement. In this case, the auctioneer has a known valuation of v for a service and the bidders face common values uncertainty regarding the cost of providing it. In a first-price reverse auction, a bidder with the lowest bid wins the auction and receives her bid as a payment from the auctioneer. The types correspond to conditionally i.i.d. signals on the cost of provision. The restriction to non-negative bids in the auction corresponds to an upper bound of v on the bids in procurement.

6.2 Second-Price Auctions

We view the first-price auction format as the natural model to describe bidding contests in situations where the set of bidders is unknown. Nevertheless, one may ask how our results would be affected if we adopt the second-price payment rule.

In the case of formal auctions nothing changes. In Chi, Murto, and Välimäki (2019), we show that in the case of two bidder types, first-price and second-price auction generate the same expected revenue. This implies that with endogenous entry, entry rates and therefore also equilibrium revenues are the same in first-price and second-price formal auctions. We should note that this equivalence relies on our finding that only two types of bidders are present in the bidding stage, as established in Lemma 5. We know from Milgrom and Weber (1982) that such equivalence would not hold if the participating bidders draw their types from a more general signal space.

In our previous working paper, we also analyzed the informal second-price auction. The main difficulty for that analysis arises from the multiplicity of symmetric bidding equilibria. However, it is clear that no matter how we select the equilibrium, the high type bidders always get a positive information rent on top of their social contribution, and hence their entry rates are distorted above the symmetric social optimum. This means that the informal first-price auction dominates the informal second-price auction in expected revenue whenever the former has no distortions.

We can further show that if we select the bidder optimal equilibrium in the bidding stage, then we get an unambiguous revenue ranking between the two informal auction formats: the informal first-price auction raises a higher revenue than the informal second price auction.

6.3 Asymmetric Equilibria and Participation by Invitation

In models with entry costs, one often finds that asymmetric equilibrium profiles dominate symmetric ones in terms of social efficiency. In this paper, we are motivated by economic settings where the set of potential participants is dispersed and it is not possible to achieve the ex ante co-ordination required for asymmetric equilibria.

One possible remedy could then be that the seller approaches only a subset of potential bidders and restricts participation to them only. Since the seller does not know the private information of the potential bidders, it is not clear if such a strategy would raise the expected revenues. Moreover, if only a small fraction of all potentially interested bidders are truly interested and if the seller incurs a cost of contacting the bidders, it is clearly better to rely on self-selecting mechanisms of the type that we analyze in this paper.

6.4 Affiliated Private Values

Our analysis can be extended to cover affiliated private values models as well. Matthews (1987) analyzes the effect of disclosing the number of bidders in a first-price auction. He assumes that best responses in the informal auction are monotone in signals, and that the bids in the formal auction are increasing in the number of participants. Under these assumptions, he demonstrates the payoff superiority of the informal auction.[19]

For private values models, second-price auction is the pivot mechanism, and as a consequence bidders collect their marginal contribution to the social welfare in the bidding stage. This means that the ex ante participation decisions coincide with the solution to a social planner's problem when restricted to symmetric entry profiles. Since bidders' rent is dissipated at the entry stage, the auctioneer collects the entire social surplus in revenues.

[19]Even though the assumptions seem natural, Pinkse and Tan (2005) have shown that monotonicity in the number of participants does not always hold. Similarly, monotonicity in signals can also be problematic in the informal auction and this can lead to existence problems in models with an exogenous random number of bidders.

Appendix: Omitted Proofs

Proof of Proposition 1.

Consider the problem

$$\max_{\pi \geq 0} W(\pi) = q[v(1)\left(1 - e^{-\Sigma_m \alpha_{1,m}\pi_m}\right) - c\Sigma_m \alpha_{1,m}\pi_m]$$
$$+ (1-q)\left[v(0)\left(1 - e^{-\Sigma_m \alpha_{0,m}\pi_m}\right) - c\Sigma_m \alpha_{0,m}\pi_m\right].$$

The first order conditions for interior solutions $\pi_m > 0$ to this concave problem are given by:

$$q_m v(1) e^{-\Sigma_m \alpha_{1,m}\pi_m} + (1-q_m) v(0) e^{-\Sigma_m \alpha_{0,m}\pi_m} = c, \tag{5}$$

where q_m is the posterior belief on state $\omega = 1$ after seeing signal θ_m:

$$q_m = \frac{\alpha_{1,m} q}{\alpha_{1,m} q + \alpha_{0,m}(1-q)}.$$

The q_m are strictly increasing in m by affiliation. Hence, having $\pi_m > 0$ for multiple values of m implies:

$$v(1) e^{-\Sigma_m \alpha_{1,m}\pi_m} = v(0) e^{-\Sigma_m \alpha_{0,m}\pi_m} = c.$$

Taking logarithms, we have

$$\begin{aligned} \Sigma_m \alpha_{1,m}\pi_m &= \log\left(\frac{v(1)}{c}\right), \\ \Sigma_m \alpha_{0,m}\pi_m &= \log\left(\frac{v(0)}{c}\right). \end{aligned}$$

Since we have two linear equations in M variables, the solutions are not necessarily unique. By affiliation, we have $\frac{\alpha_{1,M}}{\alpha_{0,M}} \geq \frac{\alpha_{1,m}}{\alpha_{0,m}} \geq \frac{\alpha_{1,1}}{\alpha_{0,1}}$. Hence we have a solution with $\pi_m > 0$ for at least two types only if the pair:

$$\begin{aligned} \alpha_{1,1}\pi_1 + \alpha_{1,M}\pi_M &= \log\left(\frac{v(1)}{c}\right), \\ \alpha_{0,1}\pi_1 + \alpha_{0,M}\pi_M &= \log\left(\frac{v(0)}{c}\right), \end{aligned}$$

has a strictly positive solution. By affiliation, this is the case only if:

$$\alpha_{0,M} \log\left(\frac{v(1)}{c}\right) < \alpha_{1,M} \log\left(\frac{v(0)}{c}\right). \tag{6}$$

If this condition is satisfied, a solution $(\pi_1^*, 0, ..., 0, \pi_M^*)$ with $\pi_M^* > \pi_1^* > 0$ to this system exists. Other positive solutions to this problem may also exist, but the aggregate amount of entry, $\sum_m \lambda_{i,m} = \sum_m \alpha_{i,m}\pi_m$, is uniquely determined for both states i across all solutions.

From equation (6), we can compute a threshold $\overline{c}$ such that positive entry by multiple types takes place if the entry cost is below $\overline{c}$:

$$\overline{c} = e^{\frac{\alpha_{1,M}\log(v(0))-\alpha_{0,M}\log(v(1))}{\alpha_{1,M}-\alpha_{0,M}}} > 0.$$

For $c \in (0, \overline{c})$ we have an interior solution with $\pi_M^{opt} > \pi_1^{opt} > 0$.

When $c \geq \overline{c}$, we have a corner solution where $\pi_m^{opt} = 0$ for all $m < M$. In that case, the optimal entry rate for the θ_M types is solved from

$$q_M v(1) e^{-\alpha_{1,M}\pi_M} + (1-q_M) v(0) e^{-\alpha_{0,M}\pi_M} = c.$$

■

Proof of Lemma 2. If $\left(\overline{b}, (\overline{\pi}_1, \overline{\pi}_M)\right)$ is an equilibrium of the reduced model (i.e. the model with $\theta \in \{\theta_1, \theta_M\}$), then types $\theta = 1$ and $\theta = M$ get the same payoff in the full model under strategy profile $\left(\overline{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$ and hence their behavior remains consistent with equilibrium. We have to prove that no other type $\theta_m \in \{\theta_2, ..., \theta_{M-1}\}$ has a strict incentive to enter against $\left(\overline{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$. Denote by $\overline{R}_0(p)$ and $\overline{R}_1(p)$ the expected payoff of a bidder who bids p against $\left(\overline{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$, conditional on state $\omega = 0$ and $\omega = 1$, respectively. Suppose that there is a type $\theta_m \in \{\theta_2, ..., \theta_{M-1}\}$, and a bid p_m such that θ_m gets strictly more than c when the others follow $\left(\overline{b}, (\overline{\pi}_1, 0, ..., 0, \overline{\pi}_M)\right)$. This means that $q_m \overline{R}_1(p_m) + (1-q_m)\overline{R}_0(p_m) > c$. But since $q_0 < q_m < q_M$, then either $q_1\overline{R}_1(p_m) + (1-q_0)\overline{R}_0(p_m) > c$ or $q_M\overline{R}_1(p_m) + (1-q_M)\overline{R}_0(p_m) > c$, implying that either type θ_1 or θ_M would get a payoff strictly greater than c by following bidding strategy p_m. This contradicts the fact that $\left(\overline{b}, (\overline{\pi}_1, \overline{\pi}_M)\right)$ is an equilibrium of the reduced model.

To prove the second part of the lemma, suppose that there is an equilibrium (b, π) in the full model such that $\pi_m > 0$ for some $m \in \{2, ..., M-1\}$. Let $R_\omega(p)$ denote the expected payoff of bidding p conditional on state ω:

$$\begin{aligned} R_1(p) &= e^{-\sum_{m=1}^{M}\alpha_{1,m}\pi_m(1-F_m(p))}(v(1)-p), \\ R_0(p) &= e^{-\sum_{m=1}^{M}\alpha_{0,m}\pi_m(1-F_m(p))}(v(0)-p). \end{aligned}$$

Since (b, π) is an equilibrium, we must have for every $p \geq 0$,

$$q_m R_1(p) + (1-q_m) R_0(p) \leq c, \text{ for } m = 1, ..., M$$

and for any $p \in \text{supp}F_m$, we have

$$q_m R_1(p) + (1-q_m) R_0(p) = c.$$

These equations imply that if $p \in \text{supp} F_m$, $m \in \{2, ..., M-1\}$, then

$$R_1(p) = R_0(p) = c,$$

and hence any type will be indifferent between entering and bidding p and staying out. It follows that if there is a strategy profile $\left(\bar{b}, (\bar{\pi}_1, \bar{\pi}_M)\right)$ in the reduced model that induces the same state-by-state payoffs $R_1(p)$ and $R_0(p)$ than (b, π), then $\left(\bar{b}, (\bar{\pi}_1, \bar{\pi}_M)\right)$ is an equilibrium of the reduced model. It remains to show that we can construct such a strategy profile.

Since $\frac{\alpha_{1,1}}{\alpha_{0,1}} < ... < \frac{\alpha_{1,M}}{\alpha_{0,M}}$, we can always choose $\widehat{\pi}_1 > 0$ and $\widehat{\pi}_M > 0$ such that

$$\begin{aligned}
\alpha_{1,1}\widehat{\pi}_1 + \alpha_{1,M}\widehat{\pi}_M &= \sum_{m=1}^{M} \alpha_{1,m}\pi_m, \\
\alpha_{0,1}\widehat{\pi}_1 + \alpha_{0,M}\widehat{\pi}_M &= \sum_{m=1}^{M} \alpha_{0,m}\pi_m.
\end{aligned}$$

Similarly, we choose c.d.f.:s $\widehat{F}_1(p)$ and $\widehat{F}_M(p)$ such that for all $p \geq 0$, we have

$$\begin{aligned}
\alpha_{1,1}\widehat{\pi}_1\left(1 - \widehat{F}_1(p)\right) + \alpha_{1,M}\widehat{\pi}_M\left(1 - \widehat{F}_M(p)\right) &= \sum_{m=1}^{M} \alpha_{1,m}\pi_m\left(1 - F_m(p)\right), \\
\alpha_{0,1}\widehat{\pi}_1\left(1 - \widehat{F}_1(p)\right) + \alpha_{0,M}\widehat{\pi}_M\left(1 - \widehat{F}_M(p)\right) &= \sum_{m=1}^{M} \alpha_{0,m}\pi_m\left(1 - F_m(p)\right).
\end{aligned}$$

We now have a strategy profile of the reduced model, $\left(\left\{\widehat{F}_1(\cdot), \widehat{F}_M(\cdot)\right\}, \{\widehat{\pi}_1, \widehat{\pi}_M\}\right)$, with the same state-by-state payoffs as the original equilibrium of the full model:

$$\begin{aligned}
\widehat{R}_1(p) &:= e^{-\alpha_{1,1}\widehat{\pi}_1\left(1-\widehat{F}_1(p)\right) - \alpha_{1,M}\widehat{\pi}_M\left(1-\widehat{F}_M(p)\right)}\left(v(1) - p\right) = R_1(p), \\
\widehat{R}_0(p) &:= e^{-\alpha_{0,1}\widehat{\pi}_1\left(1-\widehat{F}_1(p)\right) - \alpha_{0,M}\widehat{\pi}_M\left(1-\widehat{F}_M(p)\right)}\left(v(0) - p\right) = R_0(p).
\end{aligned}$$

■

Proof of Lemma 3. Let F_l and F_h denote the bid distributions of an atomless bidding equilibrium b corresponding to entry profile $\pi = (\pi_h, \pi_l)$ and let $\lambda_{\omega,\theta} = \alpha_{\omega,\theta}\pi_\theta$, $\omega \in \{0, 1\}$, $\theta \in \{h, l\}$. We start by ruling out the possibility of gaps in the union of the bid supports. This is by a standard argument: suppose that there is some interval (p, p'') such that no type bids within that interval, but p'' is in the bid support for type θ. Then type θ could reduce her bid from p'' without decreasing her winning probability and this would strictly increase her expected payoff. It follows that:

Claim 1

$$suppF_l \cup suppF_h = [0, p^{\max}] \;\; \textit{for some } p^{\max} > 0.$$

Note that this does not rule out gaps in the bid supports of individual types.

We next investigate the possible configurations of the bid supports $suppF_l$ and $suppF_h$. We first specify the range of bids where the supports may overlap, i.e. where both types can potentially be indifferent simultaneously. Denote by $U_\theta(p)$ the payoff of type θ who bids p, when bidding distributions are given by $F_\theta(p)$:

$$\begin{aligned} U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))-\lambda_{1,l}(1-F_l(p))}(v(1)-p) \\ &\quad + (1-q_\theta) e^{-\lambda_{0,h}(1-F_h(p))-\lambda_{0,l}(1-F_l(p))}(v(0)-p). \end{aligned} \tag{7}$$

Differentiating with respect to p, we have:

$$\begin{aligned} U'_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))-\lambda_{1,l}(1-F_l(p))}\left[\left(F'_h(p)\lambda_{1,h}+F'_l(p)\lambda_{1,l}\right)(v(1)-p)-1\right] \\ &+ (1-q_\theta) e^{-\lambda_{0,h}(1-F_h(p))-\lambda_{0,l}(1-F_l(p))}\left[\left(F'_h(p)\lambda_{0,h}+F'_l(p)\lambda_{0,l}\right)(v(0)-p)-1\right] \end{aligned} \tag{8}$$

In equilibrium, $F'_\theta(p) > 0$ requires $U'_\theta(p) = 0$ to maintain indifference within bidding support. To analyze when this can hold, we denote the two terms in square brackets by $B(1)$ and $B(0)$:

$$\begin{aligned} B(1) &= \left(F'_h(p)\lambda_{1,h}+F'_l(p)\lambda_{1,l}\right)(v(1)-p)-1, \\ B(0) &= \left(F'_h(p)\lambda_{0,h}+F'_l(p)\lambda_{0,l}\right)(v(0)-p)-1. \end{aligned}$$

These terms are weighted in (8) by positive terms that depend on θ only through q_θ. Since $q_h > q_l$, we note that $U'_h(p)$ puts more weight on term $B(1)$ than $B(0)$, relative to $U'_l(p)$. It is immediate that for $U'_h(p)$ and $U'_l(p)$ to be zero simultaneously, it must be that $B(1) = B(0) = 0$. Since $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$, this is possible only if

$$\lambda_{0,l}(v(0)-p) > \lambda_{1,l}(v(1)-p).$$

This can hold only for low values of p. Let us define $\widetilde{p}$ as the cutoff value such that the above inequality holds for $p < \widetilde{p}$:

$$\widetilde{p} = \max\left(0, \frac{v(0)\lambda_{0,l} - v(1)\lambda_{1,l}}{\lambda_{0,l}-\lambda_{1,l}}\right).$$

(Note that we define $\widetilde{p} = 0$ if indifference is never possible). We summarize the implications of this reasoning in the following Claim. Part 1 says that the overlap of bidding supports is possible only for $p < \widetilde{p}$. Part 2 says that in any range where only low type randomizes, value of high type is U-shaped with minimum at $p = \widetilde{p}$. Part 3 says that in any range where only high type randomizes, value of low type is decreasing and hence the low type prefers lower bids.

Claim 2 *Let $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$ be given, and let $F_\theta(p)$, $\theta = h, l$, be an atomless equilibrium bidding distribution. We have:*

1. If $F'_\theta(p) > 0$ for $\theta = h, l$, then $p < \widetilde{p}$.
2. If $F'_l(p) > 0$ and $F'_h(p) = 0$, then
$$U'_h(p) \begin{cases} < 0 \text{ for } p < \widetilde{p} \\ > 0 \text{ for } p > \widetilde{p} \end{cases}.$$
3. If $F'_h(p) > 0$ and $F'_l(p) = 0$, then $U'_l(p) < 0$.

Proof. Part 1: $F'_\theta(p) > 0$ for $\theta = h, l$ requires that $B(1) = B(0) = 0$, which is only possible if $p < \widetilde{p}$. Part 2: If $F'_l(p) > 0$, then $U'_l(p) = 0$. If $F'_h(p) = 0$, then $U'_l(p) = 0$ implies that $B(1) < 0 < B(0)$ for $p < \widetilde{p}$, and $B(0) < 0 < B(1)$ for $p > \widetilde{p}$. Since $q_h > q_l$, the result follows. Part 3: If $F'_h(p) > 0$, then $U'_h(p) = 0$. If $F'_l(p) = 0$, then $U'_h(p) = 0$ implies that $B(0) < 0 < B(1)$. Since $q_l < q_h$, the result follows. ■

With this preliminary result in place, we can prove the rest of the Lemma. Claim 1 and Part 3 of Claim 2 imply that $suppF_l = [0, p']$ for some $p' > 0$ (for the rest of the proof, we continue to use p' to denote the upper bound of the low type bid support). To see this, suppose to the contrary that there is some non-empty interval (p_1, p_2) such that $(p_1, p_2) \cap suppF_l = \emptyset$ while $p_2 \in suppF_l$. Since the union of bid supports is an interval, we then have $F'_h(p) > 0$ for $p \in (p_1, p_2)$, and by part 3 of Claim 2 we have $U'_l(p) < 0$ for $p \in (p_1, p_2)$. But then the low type has a profitable deviation from p_2 to p_1 and we have a contradiction.

We show next that if
$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} < \frac{v(1)}{v(0)}, \tag{9}$$
then the bid supports of the two types cannot overlap. Suppose to the contrary that the supports do overlap. By Part 1 of Claim 2, overlap is possible only for $p < \widetilde{p}$, and combining this with Part 2 we must have $0 \in suppF_h$ whenever the supports overlap. As before, we denote by p' the upper bound of the bid support of low types, i.e. $suppF_l = [0, p']$. Since there are no gaps in the union of the bid supports, we also have $p' \in suppF_h$. Both types

must then be indifferent between bidding 0 and p', which gives the following indifference condition:

$$\begin{aligned} & q_\theta e^{-\lambda_{1,h}-\lambda_{1,l}} v(1) + (1-q_\theta) e^{-\lambda_{0,h}-\lambda_{0,l}} v(0) \\ = \; & q_\theta e^{-(1-F_h(p'))\lambda_{1,h}} (v(1) - p') + (1-q_\theta) e^{-(1-F_h(p'))\lambda_{0,h}} (v(0) - p') \end{aligned}$$

for $\theta = l, h$. This can be rewritten as

$$\begin{aligned} 0 \; = \; & q_\theta e^{-\lambda_{1,h}} \left[\left(e^{F_h(p')\lambda_{1,h}} - e^{-\lambda_{1,l}} \right) v(1) - e^{F_h(p')\lambda_{1,h}} p' \right] \\ & + (1-q_\theta) e^{-\lambda_{0,h}} \left[\left(e^{F_h(p')\lambda_{0,h}} - e^{-\lambda_{0,l}} \right) v(0) - e^{F_h(p')\lambda_{0,h}} p' \right]. \end{aligned} \tag{10}$$

For this to hold simultaneously for both types $\theta = l, h$, both of the two terms in square-brackets must be zero. Suppose that the second term is zero, that is

$$\left[\left(e^{F_h(p')\lambda_{0,h}} - e^{-\lambda_{0,l}} \right) v(0) - e^{F_h(p')\lambda_{0,h}} p' \right] = 0.$$

Multiplying by $e^{-F_h(p')\lambda_{0,h}}$, this is equivalent to

$$\left[\left(1 - e^{-\lambda_{0,l}} e^{-F_h(p')\lambda_{0,h}} \right) v(0) - p' \right] = 0.$$

Combining this with (9) and using the fact that $e^{-F_h(p')\lambda_{1,h}} < e^{-F_h(p')\lambda_{0,h}} < 1$, this implies that

$$\left[\left(1 - e^{-\lambda_{1,l}} e^{-F_h(p')\lambda_{1,h}} \right) v(1) - p' \right] > 0,$$

and so multiplying by $e^{F_h(p')\lambda_{1,h}}$ we see that the first term in square-brackets in (10) is strictly positive:

$$\left[\left(e^{F_h(p')\lambda_{1,h}} - e^{-\lambda_{1,l}} \right) v(1) - e^{F_h(p')\lambda_{1,h}} p' \right] > 0.$$

This is a contradiction. We can conclude that it is not possible to make both types indifferent between bidding 0 and p', and so the bid supports must be non-overlapping intervals with a single point in common.

We show next that if

$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} > \frac{v(1)}{v(0)}, \tag{11}$$

then the bid supports must overlap. Suppose to the contrary that there is an equilibrium, where the bid-supports are non-overlapping intervals: $suppF_l = [0, p']$ and $suppF_h = [p', p^{\max}]$ for some $0 < p' < p^{\max}$. The low type must be indifferent between bidding 0 and p':

$$\begin{aligned} & q_l e^{-\lambda_{1,h}-\lambda_{1,l}} v(1) + (1-q_l) e^{-\lambda_{0,h}-\lambda_{0,l}} v(0) \\ = \; & q_l e^{-\lambda_{1,h}} (v(1) - p') + (1-q_l) e^{-\lambda_{0,h}} (v(0) - p'), \end{aligned}$$

which can be rewritten as

$$q_l e^{-\lambda_{1,h}}\left[\left(1-e^{-\lambda_{1,l}}\right)v(1)-p'\right]+(1-q_l)\,e^{-\lambda_{0,h}}\left[\left(1-e^{-\lambda_{0,l}}\right)v(0)-p'\right]=0.$$

For this to hold, one of the terms in square-brackets must be positive and the other one must be negative. Combining this with (11), we must have

$$\left(1-e^{-\lambda_{0,l}}\right)v(0)-p'>0>\left(1-e^{-\lambda_{1,l}}\right)v(1)-p'.$$

But then, since $q_h > q_l$, this implies that

$$q_h e^{-\lambda_{1,h}}\left[\left(1-e^{-\lambda_{1,l}}\right)v(1)-p'\right]+(1-q_h)\,e^{-\lambda_{0,h}}\left[\left(1-e^{-\lambda_{0,l}}\right)v(0)-p'\right]<0$$

or

$$\begin{aligned} & q_h e^{-\lambda_{1,h}-\lambda_{1,l}}v(1)+(1-q_h)\,e^{-\lambda_{0,h}-\lambda_{0,l}}v(0) \\ > \;& q_h e^{-\lambda_{1,h}}\left(v(1)-p'\right)+(1-q_h)\,e^{-\lambda_{0,h}}\left(v(0)-p'\right), \end{aligned}$$

and so the high type has a strictly profitable deviation to bidding 0 rather than p'. We can conclude that an equilibrium with non-overlapping intervals is not possible. The only possibility is an equilibrium with overlapping bid supports, and as we have already concluded above, in such a case 0 must be contained in the supports of both types. ■

Proof of Lemma 4. If $\pi_l^*(\pi_h)=0$, it is easy to check that (3) holds since $W(\pi_h,0)$ is concave in π_h. If $\pi_l^*(\pi_h)>0$, the first order condition for π_l must hold:

$$\begin{aligned} 0 &= \alpha_{1,l}q\left(v(1)\,e^{-\alpha_{1,h}\pi_h-\alpha_{1,l}\pi_l^*(\pi_h)}-c\right) \\ &\quad +\alpha_{0,l}(1-q)\left(v(0)\,e^{-\alpha_{0,h}\pi_h-\alpha_{0,l}\pi_l^*(\pi_h)}-c\right). \end{aligned} \tag{12}$$

If $\pi_h=\pi_h^{opt}$, then $\pi_l^*(\pi_h)=\pi_l^{opt}$ and (12) is satisfied since

$$v(1)\,e^{-\alpha_{1,h}\pi_h^{opt}-\alpha_{1,l}\pi_l^{opt}}-c=v(0)\,e^{-\alpha_{0,h}\pi_h^{opt}-\alpha_{0,l}\pi_l^{opt}}-c=0.$$

If $\pi_h>(<)\,\pi_h^{opt}$, we note that since $\alpha_{1,h}>\alpha_{0,h}$, (12) can only hold if

$$v(1)\,e^{-\alpha_{1,h}\pi_h-\alpha_{1,l}\pi_l^*(\pi_h)}-c<(>)\,0<(>)\,v(0)\,e^{-\alpha_{0,h}\pi_h-\alpha_{0,l}\pi_l^*(\pi_h)}-c.$$

But then, since $\frac{\alpha_{1,h}}{\alpha_{0,h}}>\frac{\alpha_{1,l}}{\alpha_{0,l}}$, comparison with (12) implies

$$\begin{aligned} \frac{\partial W\left(\pi_h,\pi_l^*(\pi_h)\right)}{\partial \pi_h} &= \alpha_{1,h}q\left(v(1)\,e^{-\alpha_{1,h}\pi_h-\alpha_{1,l}\pi_l^*(\pi_h)}-c\right) \\ &+ \alpha_{0,h}(1-q)\left(v(0)\,e^{-\alpha_{0,h}\pi_h-\alpha_{0,l}\pi_l^*(\pi_h)}-c\right)<(>)\,0. \end{aligned} \tag{13}$$

The result then follows from the envelope theorem. ■

Proof of Proposition 2. Lemma 3 implies that in any bidding equilibrium, the equilibrium payoff to the low types is $V_l^{IF}(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l)$. Therefore, if an equilibrium (b, π) exists, then the entry point $\pi = \left(\pi_h^{IF}, \pi_l^{IF}\right)$ must be located along the planner's reaction curve $\pi_l^{IF} = \pi_l^*\left(\pi_h^{IF}\right)$. We first note that along that curve, the following holds:

$$V_h^0\left(\pi_h, \pi_l^*\left(\pi_h\right)\right) \begin{cases} > c \text{ if } \pi_h < \pi_h^{opt} \\ = c \text{ if } \pi_h = \pi_h^{opt} \\ < c \text{ if } \pi_h > \pi_h^{opt} \end{cases}. \tag{14}$$

To see this, write the partial derivative of the planner's surplus w.r.t. π_h as:

$$\begin{aligned}\frac{\partial W\left(\pi_h, \pi_l\right)}{\partial \pi_h} &= \alpha_{1,h} q\left(v\left(1\right) e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l} - c\right) \\ &+ \alpha_{0,h}\left(1-q\right)\left(v\left(0\right) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l} - c\right) \\ &= \left(\alpha_{1,h} q + \alpha_{0,h}\left(1-q\right)\right)\cdot\Big[q_h\left(v\left(1\right) e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l} - c\right) \\ &\quad + \left(1-q_h\right)\left(v\left(0\right) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l} - c\right)\Big] \\ &= \left(\alpha_{1,h} q + \alpha_{0,h}\left(1-q\right)\right)\cdot\left(V_h^0\left(\pi_h, \pi_l\right) - c\right),\end{aligned}$$

and so the sign of $V_h^0(\pi_h, \pi_l) - c$ is the same as that of the partial of $W(\pi_h, \pi_l)$ w.r.t π_h. Hence, (14) follows from Lemma 4 (note that the partial of $W(\pi_h, \pi_l)$ w.r.t. π_l is zero along the reaction curve).

Equation (14) allows us to immediately rule out the part of the planner's reaction curve, where $\pi_h < \pi_h^{opt}$. By (14), we would have $V_h^0\left(\pi_h, \pi_l^*\left(\pi_h\right)\right) > c$ if $\pi_h < \pi_h^{opt}$, and so the high type would get strictly more than c by bidding zero. This cannot occur in equilibrium and we can conclude that in any equilibrium the entry profile must satisfy

$$\pi_h^{IF} \geq \pi_h^{opt},\ \pi_l^{IF} = \pi_l^*\left(\pi_h^{IF}\right). \tag{15}$$

We now establish the existence and uniqueness of an equilibrium. We do that separately in two different cases corresponding to the two cases presented in Lemma 3.

Fix the entry profile to the socially optimal one, and assume first that:

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} > \frac{v\left(1\right)}{v\left(0\right)} \text{ for } \pi_l = \pi_l^{opt}. \tag{16}$$

Since the left-hand side of (16) is decreasing in π_l, this implies that (16) holds for all entry profiles satisfying (15) and hence by Lemma 3 zero must be included in the bid supports for both types in any candidate for a bidding equilibrium. But since $V_h^0\left(\pi_h, \pi_l^*\left(\pi_h\right)\right) < c$

whenever $\pi_h > \pi_h^{opt}$ by (14), the high type bidder will get strictly less than c if $\pi_h > \pi_h^{opt}$. The only possibility is that the equilibrium entry profile is socially optimal and zero is in the bid supports of both type of bidders. We now establish existence and uniqueness of such an equilibrium by constructing one. Since we use necessary conditions for equilibrium in the construction and since the construction results in a unique equilibrium, this also establishes uniqueness of equilibrium under (16).

Assume that entry rates are socially optimal i.e. given by

$$\lambda_{\omega,\theta} = \alpha_{\omega,\theta}\pi_\theta^{opt}, \ \omega = 0, 1, \theta = l, h,$$

and (16) holds. Given these entry rates, define $p^{\max}$ such that a high type bidder just breaks even by bidding $p^{\max}$ if she wins for sure:

$$p^{\max} = q_h v(1) + (1 - q_h) v(0) - c.$$

Clearly, $p^{\max}$ is the only possible upper bound of $suppF_h$ in equilibrium. We construct an equilibrium by proceeding downwards from $p^{\max}$ assuming that only the high type places bids near $p^{\max}$. We get $F_h(p)$ from the requirement that high types recoup the cost of entry for all such p:

$$\begin{aligned} U_h(p) &= q_h e^{-\lambda_{1,h}(1-F_h(p))}(v(1) - p) \\ &\quad + (1 - q_h) e^{-\lambda_{0,h}(1-F_h(p))}(v(0) - p) \\ &= c \text{ for } p < p^{\max}. \end{aligned}$$

Note that we have $U_l(p^{\max}) < c$ so that the low type has no incentive to bid near $p^{\max}$. By Part 3 of Claim 2 (in the proof of Lemma 3), we have $U_l'(p) < 0$ over this range, and as we move p downwards from $p^{\max}$ we will reach a point $\overrightarrow{p}$ such that $U_l(\overrightarrow{p}) = c$. This point is pinned down by the condition that low type is indifferent between bidding $\overrightarrow{p}$ and not participating:

$$q_l e^{-\lambda_{1,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)}(v(1) - \overrightarrow{p}) + (1 - q_l) e^{-\lambda_{0,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)}(v(0) - \overrightarrow{p}) = c.$$

We have then two different cases depending on whether $\overrightarrow{p}$ is below or above $\widetilde{p}$ defined in the proof of Lemma 3.

Case 1: $\overrightarrow{p} \leq \widetilde{p}$. We can define $F_l(p)$ and $F_h(p)$ below $\overrightarrow{p}$ so that both types are indifferent for all $p \leq \overrightarrow{p}$, that is,

$$\begin{aligned} U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))} e^{-\lambda_{1,l}(1-F_l(p))}(v(1) - p) \\ &\quad + (1 - q_\theta) e^{-\lambda_{0,h}(1-F_h(p))} e^{-\lambda_{0,l}(1-F_l(p))}(v(0) - p) \\ &= c \text{ for } 0 \leq p \leq \overrightarrow{p}, \ \theta = l, h. \end{aligned}$$

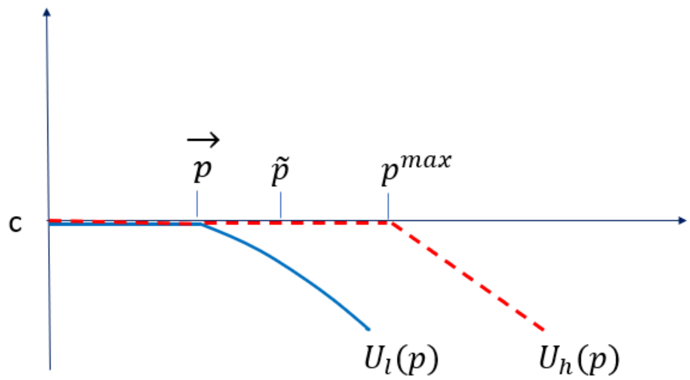

Figure 2: Value functions $U_h(p)$ and $U_l(p)$ in Case 1.

Since entry is socially optimal, the above equations automatically hold for $p = 0$, $F_l(0) = F_h(0) = 0$. As a result, we end up with an equilibrium, where the low-type bid support is $[0, \overrightarrow{p}]$ and high-type bid support is $[0, p^{\max}]$. See Figure 2 for a schematic presentation of this case.

Case 2: $\overrightarrow{p} > \widetilde{p}$. By part 1 of Claim 2, we cannot have indifference simultaneously for both types above $\widetilde{p}$, and hence the same structure as in Case 1 is not possible. Instead, we will construct an interval $[\overleftarrow{p}, \overrightarrow{p}]$ containing $\widetilde{p}$, where only the low type is active: define $F_l(p)$ within $[\overleftarrow{p}, \overrightarrow{p}]$ such that

$$\begin{aligned} U_l(p) &= q_l e^{-\lambda_{1,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)-\lambda_{1,l}(1-F_l(p))}(v(1)-p) \\ &\quad + (1-q_h)\, e^{-\lambda_{0,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)-\lambda_{0,l}(1-F_l(p))}(v(0)-p) \\ &= c \text{ for } p \in [\overleftarrow{p}, \overrightarrow{p}]. \end{aligned}$$

By part 2 of Claim 2, $U_h'(p) > 0$ for $p > \widetilde{p}$ and $U_h'(p) < 0$ for $p < \widetilde{p}$. We then define $\overleftarrow{p}$ as the point where

$$\begin{aligned} U_h\left(\overleftarrow{p}\right) &= q_l e^{-\lambda_{1,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)-\lambda_{1,l}\left(1-F_l\left(\overleftarrow{p}\right)\right)}\left(v(1)-\overleftarrow{p}\right) \\ &\quad + (1-q_h)\, e^{-\lambda_{0,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)-\lambda_{0,l}\left(1-F_l\left(\overleftarrow{p}\right)\right)}\left(v(0)-\overleftarrow{p}\right) \\ &= c. \end{aligned}$$

Since $\overleftarrow{p} < \widetilde{p}$, we can define $F_l(p)$ and $F_h(p)$ below $\overleftarrow{p}$ so that both types are indifferent, that is

$$\begin{aligned} U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))} e^{-\lambda_{1,l}(1-F_l(p))}(v(1)-p) \\ &\quad + (1-q_\theta)\, e^{-\lambda_{0,h}(1-F_h(p))} e^{-\lambda_{0,l}(1-F_l(p))}(v(0)-p) \\ &= c \text{ for } 0 \le p \le \overleftarrow{p},\ \theta = l, h. \end{aligned}$$

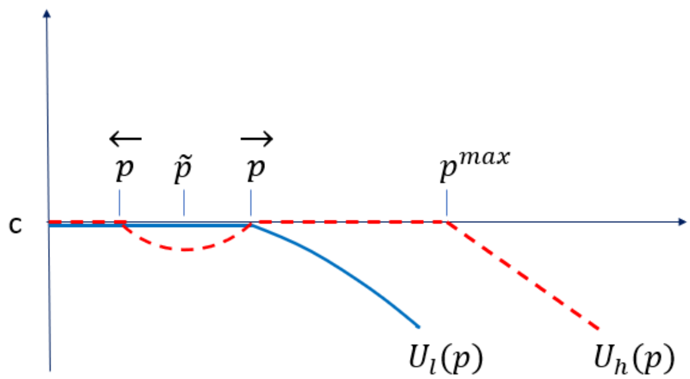

Figure 3: Value functions $U_h(p)$ and $U_l(p)$ in Case 2.

As a result we have an equilibrium, where the low type bidding support is $[0, \overrightarrow{p}]$ and high type bidding support is disconnected and given by $[0, \overleftarrow{p}] \cup [\overrightarrow{p}, p^{\max}]$. See Figure 3 for a schematic presentation of this case.

Consider next the case where

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} < \frac{v(1)}{v(0)}, \text{ for } \pi_l = \pi_l^{opt}. \tag{17}$$

Then, by Lemma 3, a bidding equilibrium with zero in the support of the high type is not possible under the socially optimal entry profile, and consequently a high type bidder would get strictly more than c under the socially optimal entry profile. A bidding equilibrium with zero in the high type support is neither possible for an entry profile $\pi_h > \pi_h^{opt}$, $\pi_l = \pi_l^*(\pi_h)$, since the high type would get strictly less than c in such a case. The only possibility is that $\pi_h > \pi_h^{opt}$, $\pi_l = \pi_l^*(\pi_h)$, and the bid supports are non-overlapping intervals with a single point in common, i.e. $suppF_l = [0, p']$ and $suppF_h = [p', p^{\max}]$ for some $0 < p' < p^{\max}$. We will now construct such an equilibrium.

Since p' is contained in the supports of both types, both types must obtain expected payoff equal to c by bidding p'. This means that expected payoff conditional on state must be equal to c in both states (otherwise the type that has a higher posterior on the state that gives a higher payoff would get strictly more than the other type). Therefore, we have:

$$\begin{aligned} e^{-\alpha_{1,h}\pi_h}(v(1) - p') &= c, \\ e^{-\alpha_{0,h}\pi_h}(v(0) - p') &= c. \end{aligned} \tag{18}$$

We now show that there is a unique pair of values p' and π_h that satisfies this pair of equations. Eliminating p' gives us:

$$e^{\alpha_{1,h}\pi_h} - e^{\alpha_{0,h}\pi_h} = \frac{v(1) - v(0)}{c}. \tag{19}$$

Since $\alpha_{1,h} > \alpha_{0,h}$, the left hand side is strictly increasing in π_h, and so there clearly exists a unique solution $\pi_h > 0$. Given this, we can solve p' from

$$p' = v(0) - e^{\alpha_{0,h}\pi_h} c.$$

Note that as long as $c < \overline{c}$, this solution always satisfies $0 < p' < v(0)$ (when $c = \overline{c}$, we have $p' = 0$). Our candidate for equilibrium entry profile is then $\left(\pi_h^{IF}, \pi_l^{IF}\right) = (\pi_h, \pi_l^* (\pi_h))$, where π_h solves (19). It is easy to check that as long as $c < \overline{c}$, we have $\pi_l^* (\pi_h) > 0$.

It is now straightforward to finish the construction of the bidding equilibrium. For $0 \leq p \leq p'$, define $F_l(p)$ so that

$$\begin{aligned} U_l(p) &= q_l e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}(1-F_l(p))\pi_l} (v(1) - p) \\ &\quad + (1 - q_l) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}(1-F_l(p))\pi_l} (v(0) - p). \\ &= c. \end{aligned}$$

Since $p' < v(0)$, there is a unique increasing $F_l(p)$ that satisfies this equation for $0 \leq p \leq p'$. Note that since $\pi_l = \pi_l^* (\pi_h)$, we have $F_l(0) = 0$ and since π_h satisfies (18), we have $F_l(p') = 1$. Similarly, for $p' < p \leq p^{\max}$, define $F_h(p)$ so that

$$\begin{aligned} U_h(p) &= q_h e^{-\alpha_{1,h}\pi_h(1-F_h(p))} (v(1) - p) \\ &\quad + (1 - q_h) e^{-\alpha_{0,h}\pi_h(1-F_h(p))} (v(0) - p) \\ &= c, \end{aligned}$$

where $p^{\max} = q_h v(1) + (1 - q_h) v(0) - c$. Again, we can check that there is a unique increasing function $F_h(p)$ that satisfies this equation for $p' \leq p \leq p^{\max}$, with $F_h(p') = 0$ and $F_h(p^{\max}) = 1$. By assumption (17) and part 2 of Claim 2, the high type has no profitable deviation to $[0, p']$, and by part 3 of Claim 2 the low type has no profitable deviation to $[p', p^{\max}]$. Obviously no type has a profitable deviation above $p^{\max}$ and hence this is an equilibrium. Since this construction is unique, the model has a unique symmetric equilibrium. ■

Proof of Lemma 5. Suppose that there is an equilibrium with $\pi_m > 0$ for some $m \in \{2, ..., M - 1\}$. Let V_ω, $\omega = 0, 1$, denote the ex-ante equilibrium payoff conditional on state for bidding according to type θ_m strategy. Since θ_m breaks just even in equilibrium, $q_m V_1 + (1 - q_m) V_0 = c$. We have $q_1 < q_m < q_M$. Hence, if $V_1 > V_0$, then type θ_M can get strictly more than c by mimicking the bidding strategy of type θ_m, and if $V_0 > V_1$, then type θ_1 can get strictly more than c by mimicking the bidding strategy of type θ_m. Therefore, it must be that $V_0 = V_1$. This of course means that any type can get c by mimicking type θ_m bidding strategy. It remains to show that some type gets strictly more by deviating from type θ_m bidding strategy.

Since we consider formal auctions, the bidding strategy is conditional on n, the realized number of entrants. Let $R_\omega^n(p)$ denote the expected payoff of bidding p, conditional on n and conditional on ω. Suppose that there is some n and either some interval (p', p'') or an atom p''' in the support of type θ_m bidding strategy such that $R_1^n(p) \neq R_0^n(p)$ within (p', p'') or at p'''. Then, either type θ_1 or type θ_M gets in expectation more than θ_m by deviating to (p', p'') or p''' for this n. By using this deviation for n, and by mimicking θ_m for all other n, this type (either θ_1 or θ_M) gets a strictly higher ex-ante payoff than θ_m, contradicting the assumption that this strategy is an equilibrium in the overall game with interim entry. It follows that unless $R_1^n(p) = R_0^n(p)$ within the whole support of type θ_m for all n, there cannot exist an equilibrium with $\pi_m > 0$, $m \in \{2, ..., M-1\}$.

It remains to show that we cannot have $R_1^n(p) = R_0^n(p)$ within the whole support of type θ_m for all n. First, note that for any $n \geq 2$, $\underline{p}_n := \min \bigcup_{i=1}^{M} \text{supp} F_i^n > v(0)$, where F_i^n is the bidding distribution of type i for realized number of entrants n. If this were not the case, then anyone bidding $\underline{p}_n$ would have a strictly profitable deviation up (recall that $n \geq 2$, so if there is no atom at $\underline{p}_n$, then one can never win by bidding there, and if there is an atom, a small deviation increases probability of winning discretely). But for all $p > v(0)$, we always have $R_0^n(p) < 0$, and hence $R_1^n(p) \neq R_0^n(p)$. ■

Proof of Proposition 3. Assume that only the highest and lowest types, h and l, enter with a strictly positive rate. Let $V_\theta^F(\pi_h, \pi_l)$ denote the expected payoff of type θ in the formal auction with entry profile $\pi = (\pi_h, \pi_l)$. We proceed to show that there is an entry profile (π_h^F, π_l^F) satisfying $V_h^F(\pi_h^F, \pi_l^F) = V_l^F(\pi_h^F, \pi_l^F) = c$. It is clear that in that case an intermediate type $\theta_m \in \{\theta_2, ..., \theta_{M-1}\}$ could not get more than c by entering. To see this, suppose to the contrary that $V_h^F(\pi_h^F, \pi_l^F) = V_l^F(\pi_h^F, \pi_l^F) = c$, but type θ_m gets strictly more than c using bidding strategy b_m against (π_h, π_l). Let V_ω denote the ex-ante payoff conditional on state ω using bidding strategy b_m. If $V_1 \geq V_0$, then type h would get strictly more than c using b_m, a contradiction. Similarly, if $V_0 \geq V_1$, then type l would get strictly more than c using b_m, a contradiction.

The unique bidding equilibrium of a formal auction is characterized in Lemma 6, and it follows that the low type gets expected payoff equal to zero whenever $n \geq 2$ whereas the high type gets strictly more than zero for any $n \geq 2$. If $n = 1$, both types get their social contribution to total welfare. Hence, for a given entry profile $\pi = (\pi_h, \pi_l)$, the low type gets $V_l^F(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l)$ and the high type gets $V_l^F(\pi_h, \pi_l) > V_l^0(\pi_h, \pi_l)$. This implies that an equilibrium entry profile must satisfy

$$\pi_h^F > \pi_h^{opt},\ \pi_l^F = \pi_l^*\left(\pi_h^{IF}\right). \tag{20}$$

It remains to show that there is a π_h^F such that $V_h^F\left(\pi_h^F, \pi_l^*\left(\pi_h^{IF}\right)\right) = c$. Note from Lemma 6 that the lower bound of the high type bidding support, $\underline{p}(n)$, as well as the probability of winning with such a bid, are continuous in (π_h, π_l). Therefore, $V_h^F(\pi_h, \pi_l^*(\pi_h))$ is continuous in π_h. We must show that $V_h^F(\pi_h, \pi_l^*(\pi_h))$ crosses c as we vary π_h. As already shown, at the efficient entry profile $\left(\pi_h^{opt}, \pi_l^*\left(\pi_h^{opt}\right)\right)$, we have $V_h^F\left(\pi_h^{opt}, \pi_l^*\left(\pi_h^{opt}\right)\right) > c$. On the other hand, when π_h is high enough, we must have $V_h^F(\pi_h, \pi_l^*(\pi_h)) < c$. To see this, note first that if only high types enter, the payoff of the high type is equal to her social contribution:

$$V^F(\pi_h, 0) = V_h^0(\pi_h, 0)\ \text{ for all } \pi_h > 0.$$

Denote by $\overline{\pi}_h$ the critical entry rate of the high type such that low types do not want to enter at all:

$$\overline{\pi}_h := \inf\left\{\pi_h : \pi_l^*(\pi_h) = 0\right\}.$$

Therefore, we have

$$\lim_{\pi_h \uparrow \overline{\pi}_h} V_h^F(\pi_h, \pi_l^*(\pi_h)) = V_h^0(\overline{\pi}_h, 0).$$

Under our assumption $c < \overline{c}$, we have $\pi_h^*(0) < \overline{\pi}_h$. Since $V_h^0(\pi_h, \pi_l)$ is strictly decreasing in π_h, it follows that $V_h^0(\overline{\pi}_h, 0) < c$, and so $V_h^0(\pi_h, \pi_l^*(\pi_h)) < c$ for π_h sufficiently high.

The result now follows from the intermediate value theorem: there must be some $\pi_h^F \in \left(\pi_h^{opt}, \overline{\pi}_h\right)$ such that $V_h^F\left(\pi_h^F, \pi_l^*\left(\pi_h^F\right)\right) = c$. Setting $\pi_l^F = \pi_l^*\left(\pi_h^F\right)$, we have $V_h^F\left(\pi_h^F, \pi_l^F\right) = V_l^F\left(\pi_h^F, \pi_l^F\right) = c$. Coupled with the bidding equilibrium characterized in Lemma 6, this is an equilibrium. Since the reaction curve $\pi_l^*(\pi_h)$ is strictly decreasing in π_h, the entry rates satisfy $\pi_h > \pi_h^{opt}$, $\pi_l^H < \pi_l^{opt}$. ■

Proof of Proposition 4. Note that

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}}$$

is decreasing in π_l with $\lim_{\pi_l \to \infty} \frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} = 1$ and $\lim_{\pi_l \to 0} \frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} = \frac{\alpha_{0,l}}{\alpha_{1,l}}$. Hence, if

$$\frac{\alpha_{0,l}}{\alpha_{1,l}} > \frac{v(1)}{v(0)}$$

and π_l is small enough, we have

$$\frac{1-e^{-\alpha_{0,l}\pi_l}}{1-e^{-\alpha_{1,l}\pi_l}} > \frac{v(1)}{v(0)},$$

and so by Lemma 3 zero is in the support of both types of bidders. In this case, the payoff in the auction is $V_\theta^0(\pi_h, \pi_l)$ and equilibrium must be socially optimal: $\pi_h = \pi_h^{opt}$, $\pi_l = \pi_l^{opt}$. This must be the case when c is sufficiently close to $\overline{c}$ (when $c = \overline{c}$, we have $\pi_l^{opt} = 0$). In the formal auction, by Proposition 3 the high type entry rate always satisfies $\pi_h^F > \pi_h^{opt}$, so that equilibrium is inefficient and generates a strictly lower total surplus than the informal auction. ■

Proof of Proposition 5. As a first step, we show that with a given entry profile (π_h, π_l), a high type gets a strictly higher payoff in the formal auction than in the informal auction if π_l is small enough.

If zero is in the bidding support of both types in the informal auction, the result is immediate. Therefore, assume that there is no overlap in the bidding supports in the informal auction and denote by p' the common point in the two supports.

Consider the following bidding strategies to compare the informal auction and the formal auction. In the informal auction, let both types bid p'. Since this is in the support of both types, it generates the equilibrium payoff to both types. In the formal auction, let both types bid $\underline{p}(n)+\varepsilon$, where ε is small. Since we can take ε arbitrarily close to zero, we calculate the winning probability and expected payoffs in the limit as $\varepsilon \to 0$. With this strategy, a bidder wins if and only if there are no (other) high type bidders, and the price in such a case is either zero or $\underline{p}(n)$. Clearly, this strategy gives the equilibrium payoff for both types (in the limit as $\varepsilon \to 0$). Given these strategies, the player gets the same allocation in both auction formats (i.e. gets the object if and only if there are no high types present), and therefore we may compare the players' payoffs across the auction formats by contrasting their expected payments conditional on winning.

Start with the low type. Since the low type gets expected payoff zero in both auction formats, she is indifferent between bidding p' in the informal auction and bidding $\underline{p}(n)+\varepsilon$ in the formal auction. Therefore, the expected payment conditional on winning must be the same in the two cases, leading to:

$$\begin{aligned} p' &= E\left(p \,\middle|\, \theta = l, \text{ "win by bidding } \underline{p}(n)+\varepsilon\text{"}\right) \\ &= \Pr\left(N^l = 0 \,\middle|\, \theta = l, N^h = 0\right) \cdot 0 \\ &\quad + \sum_{k=1}^{\infty} \Pr\left(N^l = k \,\middle|\, \theta = l, N^h = 0\right) \underline{p}(k+1). \end{aligned}$$

Consider next the high type. Her expected payment when bidding $\underline{p}(n)+\varepsilon$ in formal auction is

$$\begin{aligned} & E\left(p\left|\theta=h,\text{ ”win by bidding } \underline{p}(n)+\varepsilon\text{”}\right.\right) \\ = & \sum_{k=1}^{\infty}\Pr\left(N^{l}=k\left|\theta=h,N^{h}=0\right.\right)\underline{p}(k+1) \end{aligned}$$

and hence she prefers the formal auction if

$$\begin{aligned} & \sum_{k=1}^{\infty}\Pr\left(N^{l}=k\left|\theta=h,N^{h}=0\right.\right)\underline{p}(k+1) \\ < & \ p'=\sum_{k=1}^{\infty}\Pr\left(N^{l}=k\left|\theta=l,N^{h}=0\right.\right)\underline{p}(k+1), \end{aligned} \tag{21}$$

where

$$\underline{p}(k+1):=E\left(v\left|\theta=l,N^{h}=0,N^{l}=k\right.\right)$$

is a decreasing function of k (for $k\geq 1$). Note that the only difference in the two formulas in (21) is that the probability

$$\Pr\left(N^{l}=k\left|\theta,N^{h}=0\right.\right)$$

is conditioned on $\theta=h$ in the first line and $\theta=l$ in the second line. Since a high signal makes state $\omega=1$ more likely, a simple sufficient condition for (21) to hold is that

$$\Pr\left(N^{l}=k\left|\omega=1,N^{h}=0\right.\right)<\Pr\left(N^{l}=k\left|\omega=0,N^{h}=0\right.\right)$$

for all $k\geq 1$. Since

$$N^{l}\left|\left(\omega=1,N^{h}=0\right)\right.\sim Poisson\left(\lambda_{\omega,l}\right),$$

we know that (21) holds if

$$\lambda_{0,l}e^{-\lambda_{0,l}}>\lambda_{1,l}e^{-\lambda_{1,l}},$$

that is

$$\alpha_{0,l}\pi_{l}\,e^{-\alpha_{0,l}\pi_{l}}>\alpha_{1,l}\pi_{l}\,e^{-\alpha_{1,l}\pi_{l}}$$

or

$$\pi_{l}<\frac{\log\alpha_{0,l}-\log\alpha_{1,l}}{\alpha_{0,l}-a_{1,l}}.$$

Therefore, a high-type bidder has a lower expected payment in the formal auction for π_{l} small enough, which means that her expected payoff is higher in that auction formal.

The second step is just to conclude that changing the auction format from the informal one to the formal one will increase distortion. Fixing the equilibrium entry profile $\left(\pi_{h}^{IF},\pi_{l}^{IF}\right)$

of the informal auction, but changing the auction format to the formal auction, a high type bidder gets in the bidding equilibrium strictly more than c. To compensate this, the high-type entry rate in an equilibrium to the formal auction must satisfy $\pi_h^F > \pi_h^{IF}$. Since in both auction formats $\pi_l = \pi_l^* (\pi_h)$, the formal auction hence moves the equilibrium entry profile along $\pi_l^* (\pi_h)$ further away from the social optimum than the informal auction. By Lemma 4, the total surplus is then higher in the informal auction than in the formal auction. ■

Proof of Proposition 6. The plan of this proof is to fix the entry profile to be socially optimal, i.e. $(\pi_h, \pi_l) = \left(\pi_h^{opt}, \pi_l^{opt}\right)$, and show that there is a pair $f^* > 0$ and $-\underline{b}^* < 0$ for which the associated bidding equilibrium gives expected payoff of c to both types of entrants. Noting further that no type other than h or l would make strictly more than c, this gives the desired result. We will proceed in sequence with the formal and informal auction formats.

Formal auction:

By Proposition 2 the equilibrium entry profile is inefficient in the formal auction if and only if $c < \overline{c}$. Assume that $c < \overline{c}$, but fix the entry profile to be efficient: $(\pi_h, \pi_l) = \left(\pi_h^{opt}, \pi_l^{opt}\right)$. Since we have assumed $c < \overline{c}$, this satisfies $\pi_h^{opt} > 0$ and $\pi_l^{opt} > 0$.

Let us now equip the auction with a positive entry fee $f \geq 0$ and a negative minimum bid $-\underline{b} \leq 0$. Since we keep the entry profile fixed, the entry fee f has no effect on the bidding equilibrium, and the minimum bid $-\underline{b}$ is only binding for $n = 1$. Lemma 6 continues to hold with an obvious modification: if $n = 1$, the only participant bids $-\underline{b}$ (instead of zero). This bidding equilibrium is unique. Given this pair $(f, \underline{b})$, we denote by $V_h (f, \underline{b})$ and $V_l (f, \underline{b})$ the values of the high and low type entrant, respectively, in the unique bidding equilibrium with fixed entry profile $(\pi_h, \pi_l) = \left(\pi_h^{opt}, \pi_l^{opt}\right)$.

First, let $f = \underline{b} = 0$ and note that we have

$$V_l (0, 0) = c < V_h (0, 0) . \tag{22}$$

To see this, recall first from Section 5.1 that with socially optimal entry rates, $V_h^0 \left(\pi_h^{opt}, \pi_l^{opt}\right) = V_l^0 \left(\pi_h^{opt}, \pi_l^{opt}\right) = c$. As we can see from Lemma 6, the low type gets surplus in the bidding equilibrium if and only if there are no other entrants, and this surplus matches exactly her social value and so $V_l (0, 0) = V_l^0 \left(\pi_h^{opt}, \pi_l^{opt}\right) = c$. In contrast, the high type gets some positive surplus even when other bidders are present and so $V_h (0, 0) > V_h^0 \left(\pi_h^{opt}, \pi_l^{opt}\right) = c$.

Next, let us keep $f = 0$, but add a strictly negative minimum bid $-\underline{b} < 0$. This has no effect on the allocation and affects the payment only if no other bidders are present, in which case the payment is $-\underline{b}$ instead of zero. Recalling that we denote the number of "other" entrants of type θ by N^θ, we can write the value of type θ as:

$$V_\theta (0, \underline{b}) = V_\theta (0, 0) + \Pr \left(N^l = N^h = 0 \,|\theta \right) \cdot \underline{b}. \tag{23}$$

With socially optimal entry rates the low type believes that the probability of facing no competition is higher than what the high type believes:

$$
\begin{aligned}
\Pr\left(N^l = N^h = 0 \,|\theta = l\right) &= q_l e^{-\alpha_{1,h}\pi_h^{opt} - \alpha_{1,l}\pi_l^{opt}} + (1 - q_l)\, e^{-\alpha_{0,h}\pi_h^{opt} - \alpha_{0,l}\pi_l^{opt}} \\
&> \\
\Pr\left(N^l = N^h = 0 \,|\theta = h\right) &= q_h e^{-\alpha_{1,h}\pi_h^{opt} - \alpha_{1,l}\pi_l^{opt}} + (1 - q_h)\, e^{-\alpha_{0,h}\pi_h^{opt} - \alpha_{0,l}\pi_l^{opt}},
\end{aligned}
$$

and so it follows from (22) and (23) that there is some $-\underline{b}^* < 0$ such that $V_l(0, \underline{b}^*) = V_h(0, \underline{b}^*) = c^* > c$.

Finally, let us add an entry fee $f > 0$. Since this will not affect bidding behavior in any way, we have

$$V_\theta(f, \underline{b}) = V_\theta(0, \underline{b}) - f,\ \theta = h, l.$$

Setting entry fee $f^* = c^* - c$ then results in

$$V_h(f^*, \underline{b}^*) = V_l(f^*, \underline{b}^*) = c.$$

We have now shown that given $(f^*, \underline{b}^*)$ and entry profile $\left(\pi_h^{opt}, \pi_l^{opt}\right)$, the extreme types h and l just break even in the resulting bidding equilibrium. The only thing that remains is to note that no interior type $m \in \{2, ..., M-1\}$ can do strictly better than that. Following the logic of the proof of Lemma 5, either type h or type l gets at least the same as an interior type by mimicking the bidding behavior of that type. This rules out the possibility that an interior type gest strictly more than both h and l.

Informal auction:

Combining Lemma 3 and Proposition 1, the informal auction has an equilibrium with socially optimal entry profile $\left(\pi_h^{opt}, \pi_l^{opt}\right)$ if and only if

$$\frac{1 - e^{-\alpha_{0,l}\pi_l^{opt}}}{1 - e^{-\alpha_{1,l}\pi_l^{opt}}} > \frac{v(1)}{v(0)}. \tag{24}$$

Let us therefore fix the entry profile to be $\left(\pi_h^{opt}, \pi_l^{opt}\right)$, but assume that (24) does *not* hold. By Lemma 3 the bidding equilibrium must then consist of two non-overlapping intervals.

We then add a positive entry fee $f \geq 0$ and a negative minimum bid $-\underline{b} \leq 0$. The entry fee has no effect on the bidding equilibrium. To understand the effect of the negative minimum bid $-\underline{b}$, note that it just shifts the feasible range of bids from $[0, \infty)$ to $[-\underline{b}, \infty)$ and is therefore equivalent from the bidders' perspective to shifting bidding strategies by $-\underline{b}$ and simultaneously replacing valuations $v(\omega)$ by $v(\omega) + \underline{b}$. We conclude that for a fixed entry profile, the equilibrium bidding distribution of an informal auction with a minimum bid $-\underline{b} < 0$ is given by $F(b) = G(b + \underline{b})$, where $G(\cdot)$ is the equilibrium bidding distribution of an informal auction without reserve price and valuations $v(\omega) + \underline{b}$, $\omega = 0, 1$. Utilizing this transformation, we can show that the informal auction with exogenous entry rate $\left(\pi_h^{opt}, \pi_l^{opt}\right)$ has a unique bidding equilibrium with non-overlapping supports as long as

$$\frac{1 - e^{-\alpha_{0,l}\pi_l^{opt}}}{1 - e^{-\alpha_{1,l}\pi_l^{opt}}} < \frac{v(1) + \underline{b}}{v(0) + \underline{b}}, \tag{25}$$

while if this does not hold, the bidding supports for both types must contain zero. Since the left hand side of (25) is strictly larger than one and the right hand side is decreasing in $\underline{b}$ and converges to one as $\underline{b} \to \infty$, there is some $\overline{b} > 0$ such that for $\underline{b} = \overline{b}$ the left and the right hand sides are equal. We conclude that a unique bidding equilibrium with non-overlapping supports exists for all $\underline{b} \in \left[0, \overline{b}\right)$.

Denote by $V_l(f, \underline{b})$ and $V_h(f, \underline{b})$ the values of the two types in the unique bidding equilibrium with entry profile $\left(\pi_h^{opt}, \pi_l^{opt}\right)$, entry fee f, and minimum bid $-\underline{b}$, where $0 \leq \underline{b} \leq \overline{b}$. By the properties of the bidding equilibrium, these must be continuous in $\underline{b}$ for $0 \leq \underline{b} \leq \overline{b}$. Since entry fee has no effect on bidding, $V_\theta(f, \underline{b}) = V_\theta(0, \underline{b}) - f$, $\theta = h, l$.

For $\underline{b} = \overline{b}$, (25) is violated and bidding zero must be a best-response for both types. Therefore we can write:

$$\begin{aligned} V_\theta\left(0, \overline{b}\right) &= \Pr\left(N^l = N^h = 0 \,|\theta\right)\left[\mathbb{E}\left(v \,\middle|N^l = N^h = 0, \theta\right) + \overline{b}\right] \\ &= c + \Pr\left(N^l = N^h = 0 \,|\theta\right) \cdot \overline{b}, \end{aligned}$$

where we have used $\Pr\left(N^l = N^h = 0 \,|\theta\right) \mathbb{E}\left(v \,\middle|N^l = N^h = 0, \theta\right) = c$ due to social optimality of the entry profile $\left(\pi_h^{opt}, \pi_l^{opt}\right)$. Since $\Pr\left(N^l = N^h = 0 \,|\theta = l\right) > \Pr\left(N^l = N^h = 0 \,|\theta = h\right)$, we have $V_l\left(0, \overline{b}\right) > V_h\left(0, \overline{b}\right) > c$. On the other hand, when $\underline{b} = 0$, bidding zero is optimal only the low type, and as in the case of a formal auction, we must have $V_l(0, 0) = V_l^0\left(\pi_h^{opt}, \pi_l^{opt}\right) = c$ and $V_h(0, 0) > V_h^0\left(\pi_h^{opt}, \pi_l^{opt}\right) = c$. By continuity, we then must have some $\underline{b}^* \in \left(0, \overline{b}\right)$ such that $V_l(0, \underline{b}^*) = V_h(0, \underline{b}^*) = c^* > c$.

The rest of the proof is identical to the case of a formal auction: adding an entry fee $f^* = c^* - c$ results in

$$V_h(f^*, \underline{b}^*) = V_l(f^*, \underline{b}^*) = c,$$

and hence the pair $(f^*, \underline{b}^*)$ yieds and equilibrium with socially optimal entry. ■

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